MATH PROBE

9

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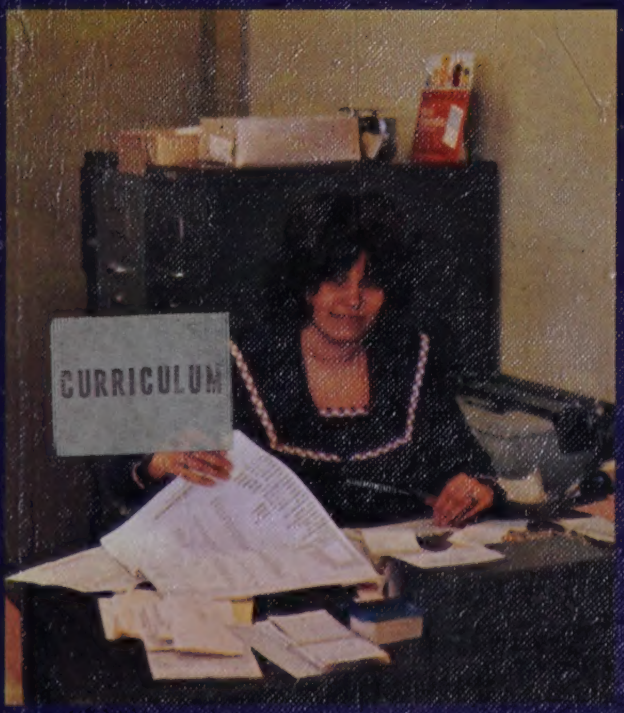

Math Probe
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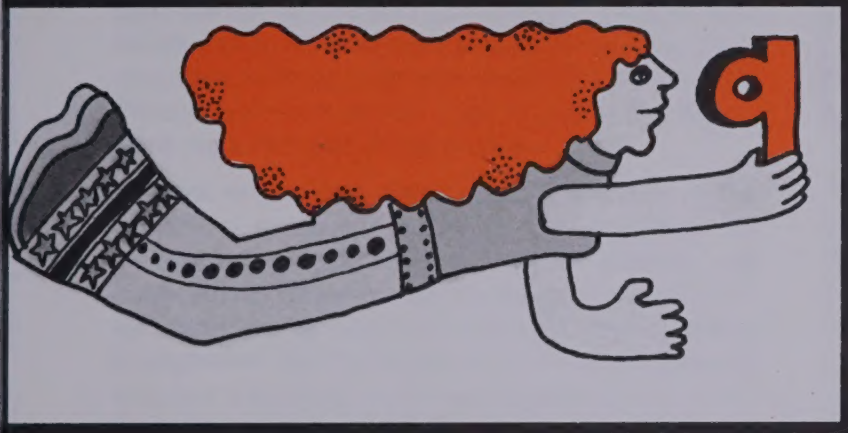
KRYSAK BYE ELLIOTT



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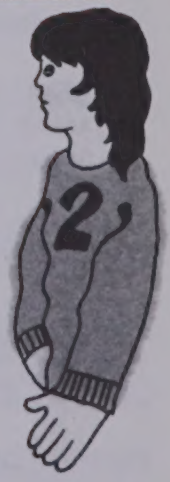

Math Probe 4

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A Note From the Authors

To The Student:

This book is written to you – the student. A number of questions are asked just as if we were talking to you – and often no answers are given – immediately. You should think about the questions, try to answer them, discuss them, then read on.

There are several features of note in this book. It is intended that the Projects be done by groups of two to five students working cooperatively, each doing his part, then sharing the results. In most instances the Projects are to provide experiences upon which to build the formal concepts later. In some cases you will be able to do the exercises without further help. One important point: you can learn much by discussing the Projects with each other and your teacher. The discussions are a vital part of this program.

No book is complete within itself. This is true of this book. In a number of instances it is suggested the class write letters to obtain information, booklets, and publications. This should be done ahead of time to ensure the material is on hand when it is needed. Also it is suggested you invite people from your community into your classroom to discuss certain topics. At other times it is suggested that you visit the community and business world. In these ways mathematics is interwoven with the community and with meaningful aspects of living – it becomes relevant.

"Using the Library" sections in this book differ from those in **Math Probe 2** and **3**. A number of books of value in a school library are listed here. Along with encyclopedia, these should provide interesting and relevant material for the student.

The exercises in "Using the Library" are essentially of two kinds. Some of the exercises are related to the construction of a time line. The class, working together, share the responsibility of putting on the bulletin board (or elsewhere in the classroom) a time line, and of researching the date of birth of each person listed and an interesting item about that person (not all the people are mathematicians). This brief information is then placed on the time line along with the suggested milestones in history.

Alternatively, the students may each construct a time line over several pages of their books and enter the names, events and interesting items thereon. The second type of assignment requires the student to research certain topics from available resources.

Suggested titles:

Aaboe, Asger. *Episodes From the Early History of Mathematics*. L. W. Singer Co. 1964.

Eves, Howard. *An Introduction to the History of Mathematics*. Holt, Rinehart and Winston of Canada, Ltd. 1969.

Newman, James R. *The World of Mathematics. 4 Volumes*. Simon and Schuster. 1956.

Northrop, Eugene P. *Riddles in Mathematics*. Pelican Books. 1963.

Pedoe, Dan. *The Gentle Art of Mathematics*. Pelican Books. 1963.

Sawyer, W. W. *Mathematicians Delight*. Pelican Books. 1943.

Sawyer, W. W. *Prelude to Mathematics*. Pelican Books. 1955.

Scientific American. *Mathematics in the Modern World*. W. H. Freeman and Co. 1968.

Smith, D. E. *History of Mathematics. 2 Volumes*. Dover Publications, Inc. 1958.

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A number of topics from **Math Probe 2** and **3** are repeated in this book, but are now presented more formally and in greater depth. If you have covered a section before, a brief reading with a practice review of some of the questions may suffice. The other sections under the same concept, only covered in more depth, will require more extensive study. You may find it interesting to obtain copies of **Math Probe 2** and **3** in order to do some of the projects listed in repeated sections.

In this book there are assignments titled "Your Job and Mathematics". These have been suggested by various business firms, employers and managers of private businesses. Each is designed by the people contacted to give you an idea of how you might be expected to use mathematics in the various vocations. We hope you will enjoy probing mathematics.

W.P.K.
M.P.B.
H.A.E.

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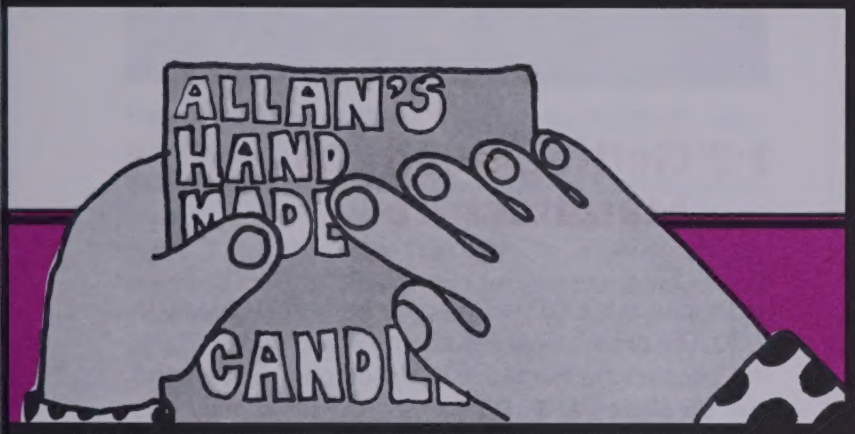
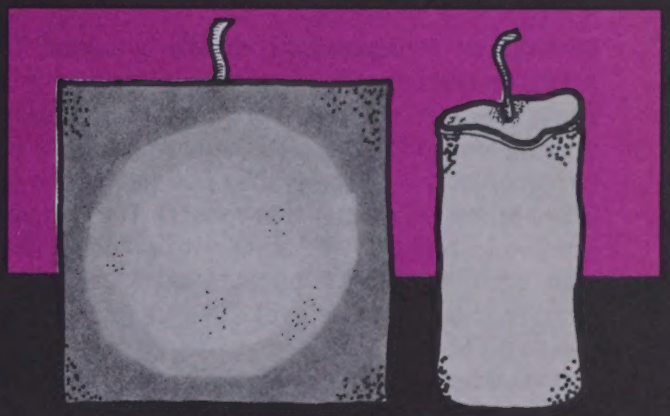
Chapter One

OPERATING A MANUFACTURING BUSINESS





1·1 Introduction

An organization that has grown rapidly in Canada since 1963 is JUNIOR ACHIEVEMENT OF CANADA. Its members are high school students who, through a practical learn-by-doing process, organize and manage their own small-scale business. Usually this business is of a manufacturing nature. In this chapter a study of manufacturing will be centred around several Junior Achievement companies that were in actual operation and made a profit.

While the procedures in this chapter primarily refer to JA companies they are similar to actual full-scale companies.

Each Junior Achievement company is formed in the fall of the year. Usually 18 to 20 students form a company. They elect the board of directors, and form the working force and sales staff (employees). Early in the formation of the company they decide on what product(s) to make (or service to render). They prepare a *prospectus* and provide capital with which to operate their business through public sale of capital stock (usually at $1.00 per share). (See Chapter 2 for full details on Stocks and Bonds.) They plan the production, distribution, advertising and selling. A full set of company books and records are kept.

EXERCISE 1·1

1. Invite a Junior Achievement member or a JA advisory representative to explain how JA operates and to describe some actual Junior Achievement companies.

2. What is a company *prospectus*? What is its purpose? Endeavour to obtain a prospectus from a new company in your province or area. What type of information does it give the reader? Do you think this prospectus would achieve the purpose for which it was designed?

1·2 Getting Ready to Manufacture

Let us look at the B.D.N. Company formed in Calgary in 1970. The product was key chains which sold at $1.25. The people who formed B.D.N. Company issued and sold 95 shares at $1.00 per share. This money was used to finance the operation until an income from sales could be realized.

In order to manufacture the key chains certain equipment and raw materials were necessary. An account was kept of the equipment. Sometimes equipment is shown as a part of *Factory Expenses* (when these costs are small).

B.D.N. Company
Factory Expenses
October 1, 1970

Qty.	Item	Price per unit	Total
1	stapler	$3.50	$3.50
3	metal files	1.25	3.75
2	safety-gloves	4.26	8.52
1	pliers	2.98	2.98
		Total	$18.75

Raw materials (those items consumed) were purchased. The cost of these items was recorded at the end of the month (the plant had not started operating yet) in the *Inventory of Raw Materials.* It should be noted here that raw materials is a term usually reserved for those items that go into the product and become a part of the product. For simplification here we will consider all consumable materials used in the process as raw materials.

B.D.N. Company
Inventory of Raw Materials
October 1, 1970

Qty.	Item	Price per unit	Total
100	moulds	$ .04	$ 4.00
1 gal	resin	6.50	6.50
1 gal	catalyst	4.00	4.00
5 pkg.	dye	1.32	6.60
100	corks	1.22/100	1.22
100	pins	.59/100	.59
100	caps	2.10/100	2.10
100	plastic bags	.42/100	.42
100	JA seals	.35/100	.35
100	key chains	2.00/100	2.00
1 pkg.	sandpaper	.65	.65
		Total	$28.43

At the end of October the *Inventory of Raw Materials* was again completed. There were certain purchases of raw materials during the month.

B.D.N. Company
Inventory of Raw Materials
October 31, 1970

Qty.	Item	Price per unit	Total
14	moulds	$ .04	$ .56
.5 gal	resin	6.50	3.25
.5 gal	catalyst	4.00	2.00
2 pkg.	dye	1.32	2.64
20	corks	1.22/100	.24
50	pins	.59/100	.30
25	caps	2.10/100	.53
25	plastic bags	.42/100	.11
100	JA seals	.35/100	.35
35	key chains	2.00/100	.70
1 pkg.	sandpaper	.65/pkg	.65
		Total	$11.33

At the end of October a *Statement of Raw Materials Used* was prepared.

B.D.N. Company
Statement of Raw Materials Used
For Month Ending October 31, 1970

Inventory of Raw Materials Oct. 1, 1970	$28.43
Purchase of Raw Materials	34.67
Shipping Charges	1.52
	$64.62
Inventory of Raw Materials Oct. 31, 1970	$11.33
Cost of Raw Materials Used	$53.29

You will note that the cost of raw materials used each month is found by *adding* the cost of raw materials at beginning of month and the cost of raw materials purchased during the month *less* the raw materials on hand at the end of the month.

In some large operations the cost of freight and some other expenses are *apportioned* out to the cost of each item. We shall not do this in this study.

EXERCISE 1·2

Calculate the cost of raw materials used in manufacturing during the periods shown in each.

1. Raw materials on hand Oct. 1, 1971 $56.20
   Raw materials purchased in October $87.50
   Raw materials on hand Oct. 31, 1971 $42.52
   Shipping charges on purchases $4.03

2. Raw materials on hand Jan. 1, 1971 $56,764
   Raw materials on hand Dec. 31, 1971 $43,228
   Raw materials purchased during 1971 $2,386,756
   Shipping charges on purchases $15,323

3. Raw materials purchased during 1972 $456,729
   Raw materials returned during 1972 $23,876
   Inventory of raw materials on hand January 1, 1972 $8,755
   Inventory of raw materials on hand December 31, 1972 $9,111
   Shipping charges on materials $576

4. Raw materials on hand February 1, 1971 $4,223
   Raw materials on hand January 31, 1972 $8,726
   Raw materials purchased in year $346,721
   Raw materials returned in July 1971 $2,300
   Shipping charges $2,411

5. What does the word *apportion* mean?

1·3 Break-Even Graph

The B.D.N. Company found they had to have a building in which to "manufacture". In conjunction with a number of other Junior Achievement Companies they found and rented a room in an old warehouse. The Company's share of the rent, including all the utilities, was $4.00 a month. The B.D.N. Company paid its employees at the rate of $.25 an hour (all Junior Achievement Companies pay similarily reduced wages compared to actual industry) while the president received $4.00 a month and all other elected officers received $2.00 a month. Some companies pay wages on a graduated scale as an incentive to produce more. (See Chapter 6 of **Book 2**.) Also, the B.D.N. Company pays 10% selling commission.

The B.D.N. Company experienced difficulties in production the first month. The moulds were unsatisfactory. Production was very low.

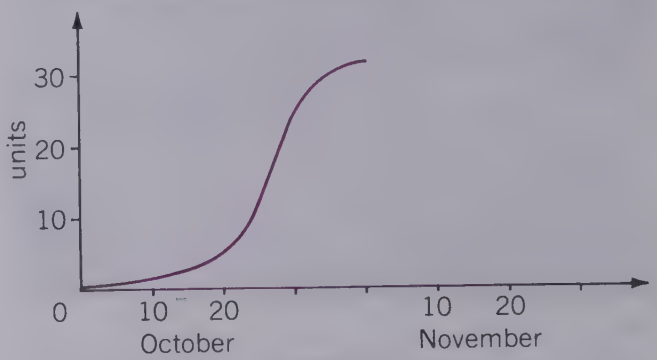


Figure 1-1

Though production was down and sales were zero, expenses over and above raw materials and equipment still continued to climb.

Expenses	
Wages	$28
Salaries	14
Overhead	5
Total	$47

Finally the problem with the moulds was solved. The company was in full production. The directors of the company wished now to establish a sales price for their products. After some preliminary calculations they made the following *Break-Even Graph.* This graph enables the directors to determine the number of key chains they must sell at a given price before it shows a profit. They established this table of estimated costs for the total operation.

Overhead for 6 months	$ 90
Salaries for 7 months	100
Wages	160
Equipment	30
Fixed cost	$380
Materials	$ .34 per unit
Commission	10%

This information was shown on a graph.

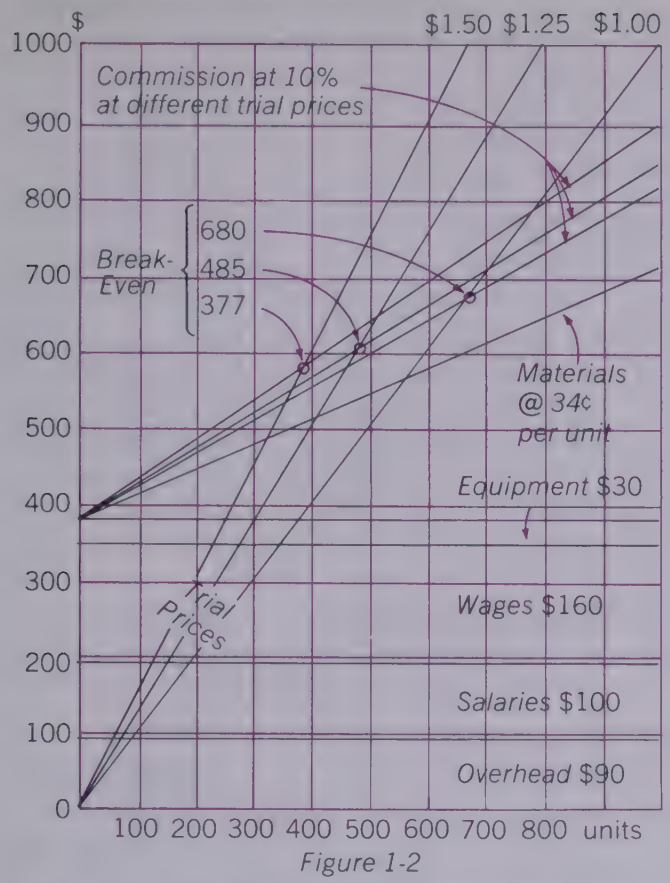


Figure 1-2

From this graph the directors observe that at $1.00 per key chain they would break even if about 680 key chains were sold. This means that if they buy just enough material for 680 key chains and sell them all they will just break even – no loss – no profit. But the directors of B.D.N. want to make a profit. If at $1.00 each they made and sold 700 their profit would be about $20 (read from the graph). Do you see how this figure is obtained?

Now the directors must decide what they think the likely sales will be and how much profit they would consider feasible. In this case the directors thought they could sell about 800. They could plainly see that at $1.00 each their profit would be small ($50 or so). With the break-even point being 680 the allowance was not sufficient.

On the other hand if they set the price at $1.50 each the break-even point would be about 377 units (from graph). The profit would be large if 800 units were sold. But would they be able to sell that many at $1.50 each? They thought this higher price might impede sales. They settled for a sales price of $1.25. (In actual practice a new company might make a number of surveys in the business community to get some indication of the public's reaction to the manufactured item and to the price.)

EXERCISE 1·3

Refer to the graph in Figure 1-2.

1. What is the break-even point for the unit price of $1.25?

2. Give a breakdown of the cost of producing and selling 377 key chains at $1.50 each.

3. Analyse the cost of producing and selling 600 key chains at $1.25 each. What profit is realized on 600 units?

4. If the B.D.N. Company sold only 400 key chains at $1.25 each, describe the financial situation of the company.

5. What proportion of the total cost of producing and selling are the salaries, wages and commission together of:
   (a) 200 key chains at $1.25 each
   (b) 500 key chains at $1.25 each
   (c) 800 key chains at $1.25 each?

   Discuss your answers. Is this what you would expect?

6. What proportion of the total cost of selling are the cost of materials and plant operation together (not salaries, wages and commission) of:
   (a) 200 key chains at $1.25 each
   (b) 500 key chains at $1.25 each
   (c) 800 key chains at $1.25 each?

   Discuss your answers. Is this what you would expect?

7. Discuss the ramifications of questions 5 and 6 to the directors of B.D.N.

8. What proportion is the profit of sales on:
   (a) 400 units at $1.25 each
   (b) 600 units at $1.25 each
   (c) 800 units at $1.25 each?

9. Describe the financial situation the company would be in if B.D.N. produced 800 units and only sold:
   (a) 570 units
   (b) 600 units
   (c) 630 units.

10. Draw a break-even graph for the company with the following estimated yearly expenses on a production and sales of 1000 units:

Overhead	$135.00
Salaries	$ 90.00
Wages	$140.00
Materials	$ .25 per unit
Commission	$ .13 per unit
Trial prices:	$1.00, $1.10, $1.25 per unit

    From this graph can you give some guiding principles to the company directors?

11. Following is the data a company has when preparing its break-even graph. Draw the graph for the company (all given for 1 year).

Overhead	$100.00
Salaries	$100.00
Wages	$160.00
Materials	$ .35 per unit
Commission	10%
Trial prices:	$1.10, $1.30, $1.50 per unit

    Estimated maximum number of units to be manufactured and sold during the year is 600. Review the break-even graph to decide whether or not this number is reasonable at the trial prices listed. Which trial price would you suggest if you were a director?

    Explain the basis for your answers.

12. Draw a break-even graph for the company with the following estimated yearly expenses on a production and sale of 400 units.

Fixed overhead	$45.00
Wages	$80.00
Salaries	$60.00
Materials	$ .20 per unit
Commission	10% on first 200, 15% on balance
Trial prices	$1.00, $1.25 per unit

    What does the chart tell the directors?

    Discuss.

1·4 Keeping Track of Current Operations

A company should keep track of its production costs, sales, profit and loss, and share values. In this section we will illustrate graphically records that the B.D.N. Company kept for the 6 months it was in operation.

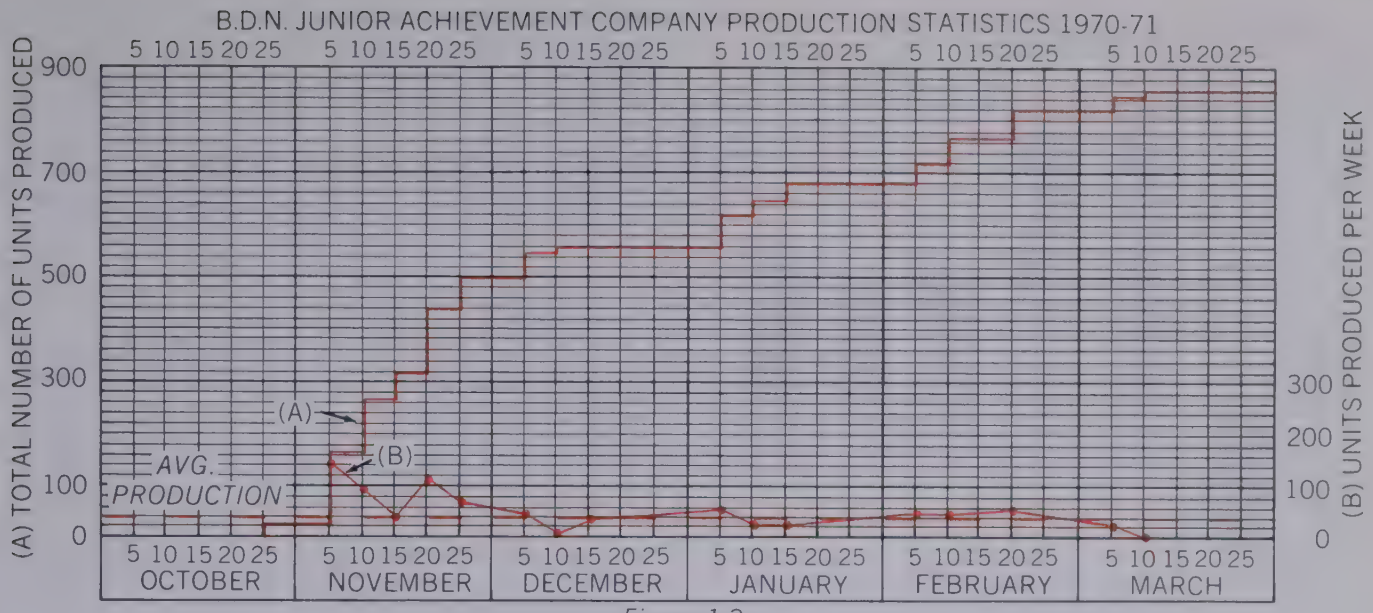


Figure 1-3

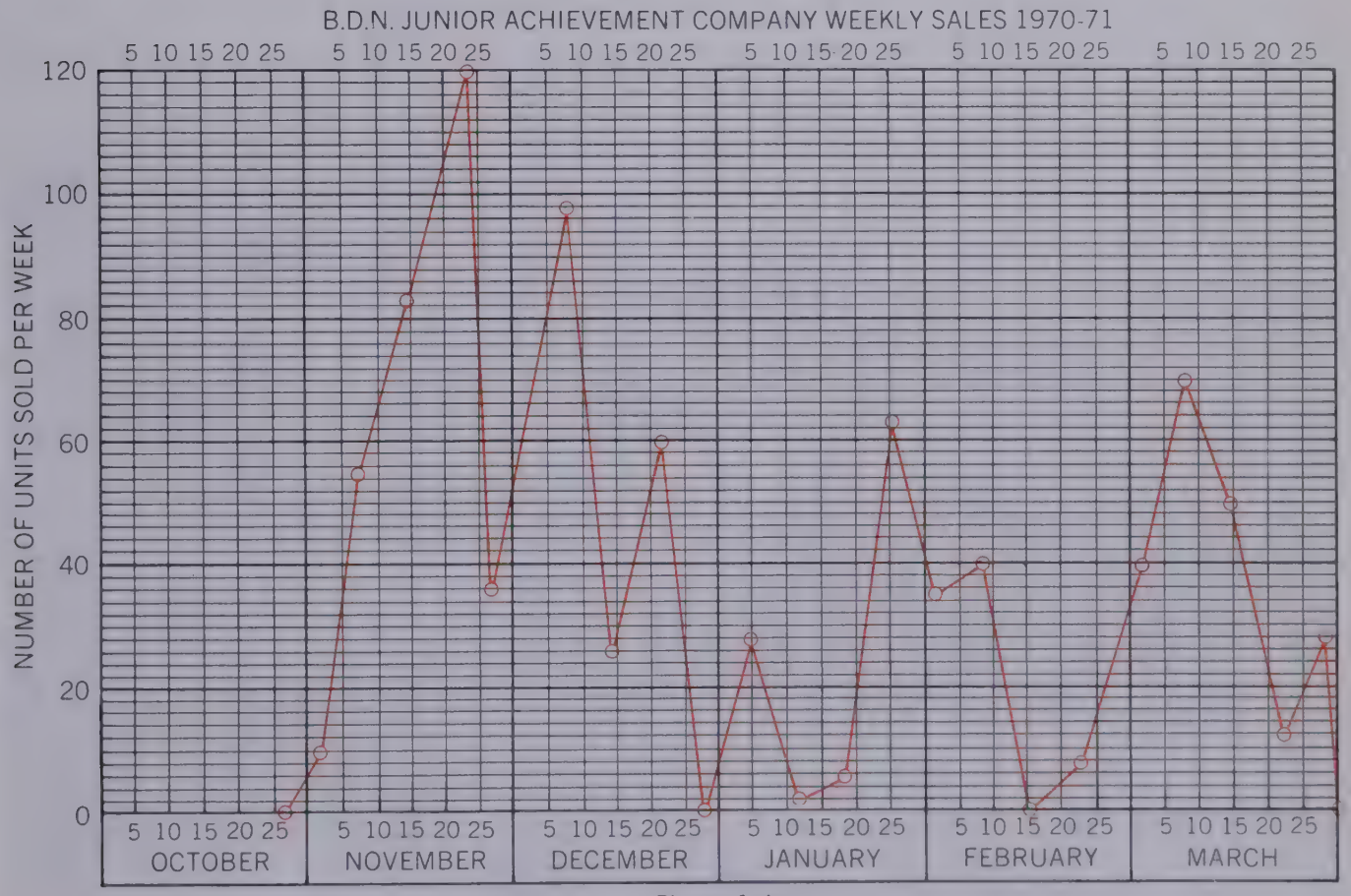


Figure 1-4

B.D.N. Company Gross Profit and Loss Report

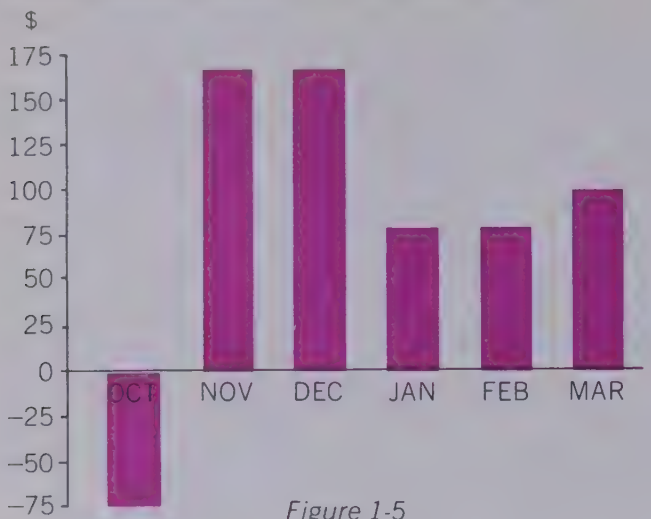


Figure 1-5

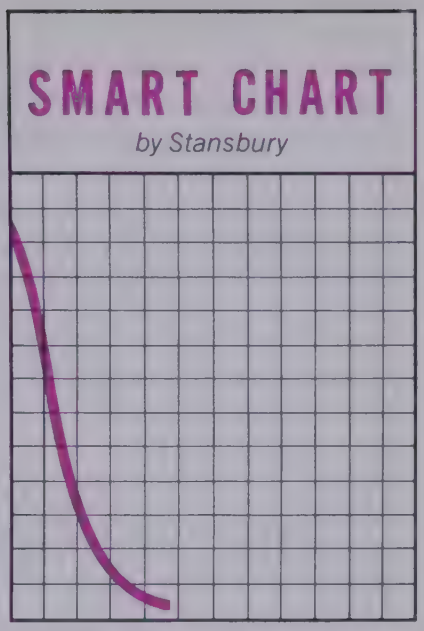
EXERCISE 1·4

1. What number of units was produced in the week ending November 20? December 5? January 15?
2. What was the total number of units produced for the week ending October 20? November 20? December 5? January 15? March 20?
3. What was the average production per week?
4. During which week was the production the highest?
5. What was the number of units sold in the week November 10-15? December 10-15? February 10-15?
6. During which period of time were the sales the highest?
7. Why do you think sales were so low the last week of December?
8. Why do you think there was a loss shown in October?
9. Sales were the highest in November, yet the gross profit was no greater than in December. Can you explain this?

   Hint! Refer to the Production Record
10. In March all raw materials on hand were *written off* as expenses. How does this explain the fact that sales were high yet profit was low?

11. From the following data construct a Production Record graph, a Sales Record, and a Gross Profit graph.

MONTH	UNITS PRODUCED			LABOUR			
	SMALL ($1.00)	MEDIUM ($1.25)	LARGE ($1.50)	TOTAL SALES	COMMISSION SALARIES	FACTORY EXPENSES	RAW MATERIALS
Oct. 1	0	0	0	$ 0	$ 0	$25	$ 25
15	0	10	5	0	65	10	25
Nov. 1	0	10	10	33	0		
15	0	100	30	19	60	4	55
Dec. 1	0	100	15	86	0		
15	0	50	0	210	30	3	10
Jan. 1	0	0	0	11			
15	50	0	0	13	55	3	5
Feb. 1	60	0	0	115			
15	140	0	0	80	40	3	40
Mar. 1	150	0	0	117			
15	20	0	0	213	65	3	35
	420	270	60	$897	$315	$51	$195



"The Chief says that our stockholders have taken their dreams elsewhere."

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1·5 Year End Reports

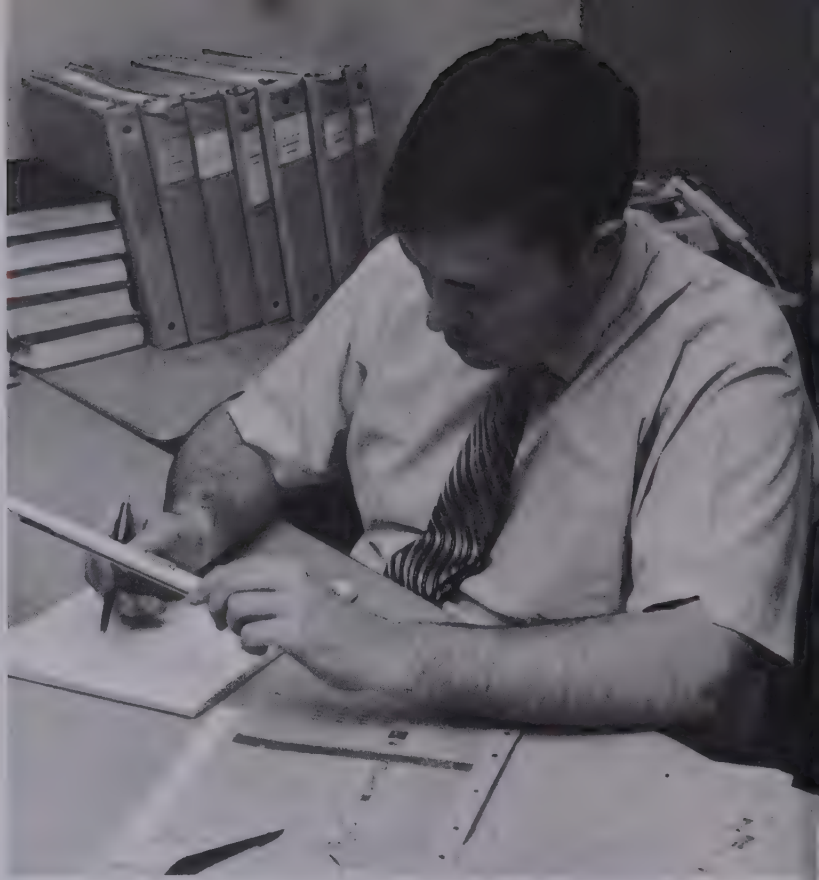
At the end of the Junior Achievement Year (usually at the end of April) each company must present a Profit and Loss Statement and Balance Sheet. This is a condensed statement of the year's operation.

Note the realistic situation under which the JA operation operates. In a regular business, company profit tax would have to be paid. All JA companies must pay a similar tax which goes to the parent Junior Achievement organization to cover operating expenses, such as awards and scholarships.

In actual business, a nominal *Federal Income Tax* of 22% of all profits after deductions and a *Surtax* of 26% on all profits in excess of $25,000 are levied. Junior Achievement sets a similar tax of 22% and 26% on profits in excess of $25.

B.D.N.
Profit and Loss Statement
For the Fiscal Period
October 5, 1970 to April 26, 1971

Sales Revenue		$1,081.50
Cost of Goods Sold		
Materials	$311.50	
Labour	123.30	434.80
Gross Profit		646.70
Operating Expenses		
Rent	$24.00	
Office Supplies	30.00	
Salaries	85.00	
Commissions	87.00	
Other	28.40	254.40
		392.30
Contribution to Achiever Trust Fund		75.00
		317.30
Cost of Report		25.00
Net Profit before Taxes		292.30
Income Taxes		140.08
Net Profit		$ 152.22
Return to Stockholders	$2.60 per share	



A company official works on sales' forecasts and margin calculation. Without these initial calculations the company might make many wrong decisions.

B.D.N.
Balance Sheet
April 26, 1971

Assets		
Cash		$ 247.22
Liabilities and Capital		
Stock – 95 @ $1.00	$ 95.00	
Net Profit	152.22	$ 247.22

Note that the B.D.N. Company had sold 95 shares at $1.00 each. The profit of $152.22 was apportioned to the 95 shares. Stockholders received $2.60 per share.

At the end of April each Junior Achievement company terminates its operation and closes out. Each company then prepares a liquidation report.

Liquidation Report

Following the normal practice of Junior Achievement Companies, B.D.N., a Junior Achievement Company, Alberta, hereby announces its formal termination as of the close of business April 26, 1971

Cash in the bank as at April 19, 1971		$487.30
From this cash we have paid these obligations:		
Contributions	$ 75.00	
Stockholders' Report	25.00	
Company Profits Tax	140.08	
Total Payments		240.08
Balance		$247.22

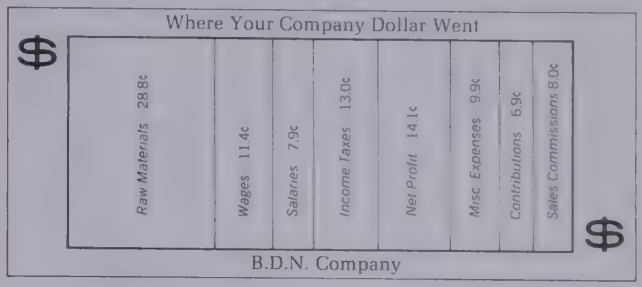
The balance was distributed to stockholders as follows:

10 cents a share as a dividend on earnings similar to that paid by businesses that continue in operation year after year	
Total dividends	$ 9.50
100 cents a share as return of original capital (par value $1.00 a share)	
Total capital	$ 95.00
150 cents a share as distribution of remaining surplus to stockholders	
Total surplus	$142.00

Summary:

Initial capitalization	$ 95.00
Total distribution to stockholders in liquidation of assets	$247.00

Another portion of the report indicates where the company's sales dollars went. The directors prepared this portion of the report to show the shareholders the business had been well managed and properly conducted. No expenses were inordinately high or out of line.



There are many other factors that go into the operation and accounting of a manufacturing business but this simplified version illustrates some of the basic business aspects. Some provinces require collection of a sales tax (see **Book 3**), payment of income taxes, (in addition to the Federal Income Tax), Employees' Workmen's Compensation Insurance, Employees' Social Insurance, Employees' Withholding Tax, Excise Tax, to name a few complicating factors.

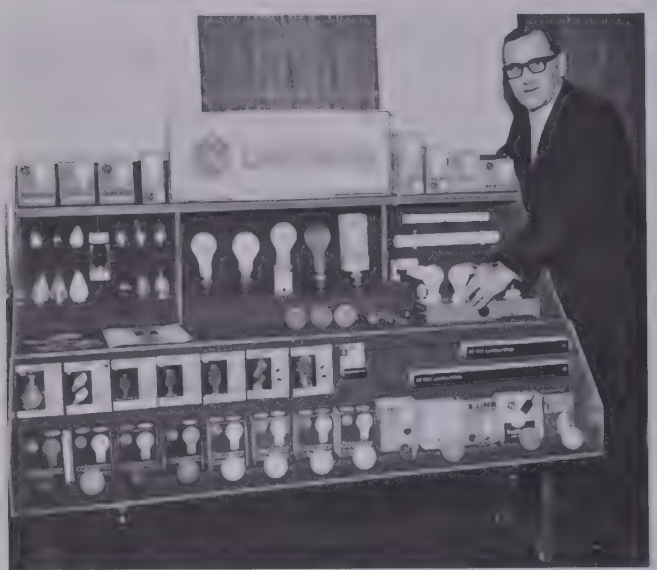
EXERCISE 1·5

1. Use a dictionary to determine what a *liquidation report* is.
2. What does *fiscal* mean as in *the fiscal period.*
3. Analyse the *Liquidation Report* of the B.D.N. Company. How did the company distribute the $247 net profit?
4. A Junior Achievement company, ABC, had sold 120 shares at $1.00 each. The net profit was $252. What amount would be returned to the shareholders? If a 10% dividend was declared and the surplus divided amongst stockholders, show the distribution schedule of the $252.
5. Use the data from the *Profit and Loss Statement* to show how the B.D.N.'s Company dollar was spent by:
   (a) a circle graph (b) a rectangle graph.

6. Show graphically how the ABC's dollar was spent from this information:

ABC COMPANY		
SALES		$900.00
COST OF GOODS SOLD		
Raw materials	$195.00	
Labour	130.00	325.00
GROSS PROFIT		575.00
EXPENSES		
Rent	$ 24.00	
Salaries	116.00	
Sales Commission	74.00	
Other Expenses	56.00	270.00
		305.00
TAXES	167.00	
LIQUIDATION INCOME	120.00	47.00
NET PROFIT		$258.00

7. In question 6, the ABC Company had sold 84 shares at $1.00 each.
   (a) Prepare a Balance Sheet.
   (b) Prepare what might be a fair distribution of the balance to its shareholders. (Suggested dividend at 10%)



The displaying and advertising of products are all important to the success of the products. What items have you purchased mainly as a result of good advertising or effective displaying?

8. Show graphically how the MNO Company's dollar was spent from this information. The company sold 97 shares at $1.00 each.

MNO COMPANY		
SALES		$1,801.00
COST OF GOODS SOLD		
Materials	$460.00	
Labour	135.00	595.00
GROSS PROFIT		1,206.00
OPERATING EXPENSES		
Rent	$ 32.00	
Office Supplies	41.00	
Salaries	160.00	
Commissions	224.00	
Other Expenses	41.00	498.00
		708.00
CONTRIBUTIONS		140.00
		568.00
COST OF REPORT		32.00
NET PROFIT BEFORE TAXES		536.00
INCOME TAXES		212.00
NET PROFIT		$324.00

9. Prepare a balance sheet and a suggested distribution schedule of the net profit. (Suggested dividend at 15%.)

Project

Project 1

Purpose

To determine what it takes to start a manufacturing company.

Method

1. Invite a Junior Achievement official to visit your class. Discuss with him part 2.

   If no Junior Achievement organization exists in your area do part 2 only. Discuss the questions within the class and with your teacher.

2. If you formed a company what product would you manufacture? Why?

   What sales potential do you think there is in your area?

   What attributes must a product have to sell well?

   What features must a manufactured item have to yield a good profit?

   Discuss the *sell a few at a high price* versus the *sell many at a low price* philosophy of getting rich.

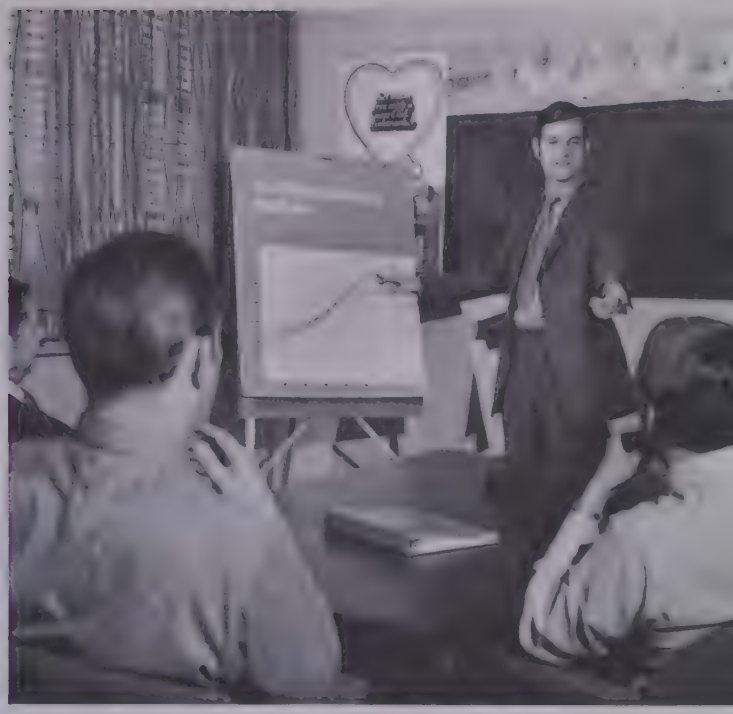
   How important is the *sales approach* in building sales?

   List the *good* attributes a salesman should have to be successful.

   List those characteristics which will likely result in a salesman not getting sales.

   List the selling points you would use in your sales pitch (sales approach) if you were selling

   (a) key chains
   (b) a game (such as Chinese Checkers)
   (c) a spring and chain for holding the car trunk lid when partly open
   (d) treated evergreen cones for producing colored flames in fireplaces
   (e) Christmas decorations
   (f) Christmas, Valentine and Easter cards.



The company officials (above) are discussing market trends in kitchen appliances with the view to long-term planning in the company's production. The packaging of the product is extremely important. Do you buy because of the package? What might you buy because of the package?

Using the Library

1. A humorous satire science fiction is Edwin A. Abbott's *Flatland* published in paperback by Dover Publications Inc., New York. Mr. Abbott has some views of women and men, aptly described in this little book.

2. *Parkinson's Law* by C. Northcote Parkinson published by Ballantine Books, New York, 1964, uses mathematical manipulation and logic to create a "humorous critique of managerial bureaucracy". Parkinson's Law can be stated simply: "Work expands so as to fill the time available for its completion."

3. Obtain and view the film *Donald Duck in Mathmagic Land* produced by Walt Disney Corp.

4. Obtain *The Mathematics Teacher* Volume LXIII, Number 4. Read the centre insert "Women of Mathematics". Why do you think there have been so few women mathematicians? Discuss this question in class.

   What effect do you think the "Women's Liberation" movement of the 1970's will have on the number of women mathematicians?

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Illustration courtesy Walt Disney Productions

REVIEW EXERCISE 1

1. Calculate the cost of raw materials used in manufacturing during the periods shown.

Raw materials on hand March 1, 1971	$5,326
Raw materials on hand February 29, 1972	$10,726
Raw materials purchased during year	$198,760
Raw materials returned July 22, 1971	$3,297
Shipping charges for year	$2,009

2. From the following data draw a break-even graph for the year's operation.

Overhead	$150.00
Salaries	$ 75.00
Wages	$ 75.00
Materials	$ .30 per unit
Commission	10% of sales
Trial prices:	$1.00, $1.50, $2.00 per unit

   Expected sales range: from 400 to 600 units

   (a) What is the break-even number of units at $1.50 per unit?

   (b) What price would you suggest if you were a director of this Junior Achievement Company? Explain why you selected the price you did.

3. Prepare a suggested distribution to stockholders at the close of business of ACE Company as of April 25, 1972.

Cash in bank as of April 24, 1972		$560.30
Disbursements:		
Stockholders Report	$ 60.00	
Company's Profit Tax	122.00	
Contributions	60.00	242.00
Net Balance		$318.30

   Initial Capitalization 120 shares @ $1.00

   Recommended dividend (by directors) 15%

Chapter Two

LET'S INVEST

2·1 Introduction

To The Students and Teacher

It is important that arrangements be made to have in your classroom during the study of this chapter:

1. A daily paper with stock market quotations. (If necessary, have it delivered by mail even if it is several days late.)
2. The Financial Post, The Financial Times or other Financial newspaper on a regular basis

In the previous chapter mention was made of a company selling shares in order to obtain working capital. Often the directors of a company do not have enough capital to start or to keep a company operating. This initial cash is used to purchase supplies, equipment, to pay wages, salaries, rent, office expenses, etc. required to produce a product or to give service.

There are people who are willing to lend money in hope of realizing a *dividend.* Corporations are privately owned by stockholders who have invested money in hopes of a profit. In a business that has been established for some time, capital includes many *assets* such as buildings, machinery, and office equipment.

Junior Achievement directors (discussed in the previous chapter) sell their shares personally to parents, neighbours, and the public. Stocks of companies such as General Motors, Gulf Oil, Scurry-Rainbow, etc. are marketed on a Stock Exchange.

Junior Achievement rules require that only one share in a company can be purchased per person. *This broadens the base of ownership and limits the stockholders' risk.* JA stock is *common stock* since all owners share equally in the profits. Also JA stock is nontransferable and may not be traded by its owners.

A copy of the *Prospectus* which states the reasons the company needs money, the uses to which the money will be put, and other facts that will encourage investments, is usually presented to each prospective buyer of shares. The salesman explains the advantages of investing in the company. A sample of a Junior Achievement *Stock Certificate* is shown here.

NON-LEGAL
ONE SHARE

PAR VALUE $1.00 ONE DOLLAR
LIMIT ONE SHARE PER PERSON

Stock Certificate

JUNIOR ACHIEVEMENT

NON-TRANSFERABLE
CERTIFICATE NUMBER
H406185
REDEEMABLE WITHIN ONE YEAR AFTER DATE OF ISSUE

DATE_____

This Certifies that

FIRST NAME INITIAL LAST NAME NUMBER STREET

CITY STATE OR PROVINCE ZIP CODE IS THE OWNER OF

ONE SHARE, PAR VALUE ONE DOLLAR, OF THE CAPITAL STOCK OF

A JUNIOR ACHIEVEMENT COMPANY IN

CITY STATE OR PROVINCE SUBJECT TO INFORMATION ON THIS CERTIFICATE

AN OFFICIAL JUNIOR ACHIEVEMENT CERTIFICATE

ACHIEVER REPRESENTATIVE STOCKHOLDER

THE STOCKHOLDER BY SIGNATURE APPOINTS THE ACHIEVER PROXY TO VOTE FOR ELECTION OF BOARD OF DIRECTORS

ORIGINAL STUB

INSTRUCTIONS
DO NOT DESTROY IF FILLED OUT WRONG. MARK SUCH CERTIFICATE VOID AND TURN IT IN.
THIS STUB IS A CHARGE AGAINST SALESMAN FOR $1 CASH OR RETURN OF UNSOLD CERTIFICATE
WHEN FORM IS ISSUED, SALESMAN FILLS IN NAME AND RETURNS STUB TO PERSON ISSUING CERTIFICATE
DO NOT TEAR OFF DUPLICATE STUB
CAREFUL - CARBON SET

ONE SHARE CERTIFICATE NUMBER H406185 $1.00
ISSUED TO

SALESMAN

DATE PAID ISSUED BY RETURNED

A Junior Achievement stock is non-legal since the company is not a legal corporation. Its members are under the legal age and it is not chartered by a province. To issue legal stock a group of promoters must apply for and obtain a charter from a provincial government. This charter authorizes the formation of a company and the right to sell shares of stock.

EXERCISE 2·1

1. What is a dividend?
2. What is a corporation? an asset? a prospectus?
3. Explain what is meant by this statement from the previous section: "This broadens the base of ownership and limits the stockholders' risk". Why do you think Junior Achievers want to *broaden the base of ownership* and why do they want to *limit the stockholders' risk?* (An Achiever is a Junior Achievement member selling the stock.)
4. What is the legal age in your province? Why do you think it may differ with the other provinces? Do you think it should differ? What is the legal age with reference to Federal matters?
5. Note the sentence on the Stock Certificate that reads "The stockholder by signature appoints the Achiever proxy to vote for election of Board of Directors." What does this sentence mean?
6. Why do you think each stock certificate is numbered?
7. The certificate states "One share, par value...". What does *par value* mean?

Project

Project 1

Purpose

To get additional material on stocks and bonds.

Method

Write Merrill Lynch, Pierce, Fenner and Smith Inc., Calgary, Montreal, Toronto, or Vancouver, requesting the following material.

1. How To Read a Financial Report
2. What is Margin?
3. Investing for Tax-Free Income
4. Growth Stocks
5. How To Buy Stocks
6. What Every Woman Ought to Know About This Stock and Bond Business

2·2 Common and Preferred Stock

Figure 2-2

Let us consider the Inkless Pen Company that has filed the necessary incorporation papers and has received its charter. The company wishes to raise $100,000. It decides to sell 10,000 shares at $10 a share. This $10 price is called the *par value* (face value). It is the initial selling price of the stock. Its price may change in subsequent selling. The company sells 10,000 shares and obtains $100,000 *capitalization* The 10,000 shares represent the *common stock* of the Inkless Pen Company. The common stock simply means that the holders of this stock receive dividends after all other commitments have been met. (Compare this with Preferred Stock later.) The stockholders, the people who bought the shares, are part owners in the company. If a man bought one share he owns $\frac{1}{10,000}$ of the company; if he bought 10 shares he owns $\frac{10}{10,000}$ or $\frac{1}{1,000}$ of the company. As evidence of his ownership, he receives a stock certificate showing the number of shares he owns.

Now let us look at the dividends on common stock. Suppose that in the first year after all the expenses have been met, the company's earnings or profit is $10,000. This is 10% of its capitalization of $100,000. The directors of the company must decide what they will do with the $10,000. They might vote to pay $5,000 in dividends and reinvest in the company the other $5,000 as *retained earnings*. Now the dividend per share is $.50.

$$\text{Percent return} = \frac{\text{dividend per share}}{\text{value paid per share}} \times 100$$

$$\% \text{ return} = \frac{.50}{10} \times 100$$

$$\% \text{ return} = 5.$$

Typical board room in a brokerage house with quotation board and desks manned by representatives who buy and sell securities for the public.

Now the $5,000 invested in the company will raise the value of each share. Originally each share was worth $10, now it might be valued at $10.50 since the company now presumably has a capital of $105,000. Now that par value figure of $10.00 a share is fictitious. As the years go by and the company prospers and expands, the capital increases and the assets of the company increase. Hence the term par value for common stocks usually lacks significance. So many companies do not set value on their stock, in which case it is known as *no-par-stock.*

How do we know what a share of stock is worth? The answer is, a share of stock is only worth what somebody is willing to pay for it when it is for sale. The price of stock, like so many other things, is determined by the supply and demand. The price of stock takes on a meaning only when you consider its earnings and dividends per share. In the case of big, long-term corporations such as General Motors, a history of its earnings and dividends is available.

If a second issue of shares is offered to raise more capital for expansion or some other reason, the company will often class the original shares Class A and the new shares Class B stock. Usually the Class B stocks carry fewer privileges than Class A. For example, Class B may not carry voting rights.

Preferred Stock

Let us return to the Inkless Pen Company after it has had a number of successful years. Its dividends have been modest. Most of the earnings have been reinvested in the company to expand production.

The directors of I.P.C. are informed that the Leadless Pencil Company, a well-known competitor, can be acquired for $75,000. The directors agree this would be a good buy but where will they get $75,000?

The directors of I.P.C. propose that they *merge* by issuing 7500 shares at par value of $10.00 per share of *preferred stock* in the name of Inkless Pen Company. These shares are then given to the owner of the Leadless Pencil Company in exchange for their company. The I.P.C. elects not to give voting rights with these *preferred stocks.*

The preferred stock would have a *fixed dividend rate.* This dividend is paid first before any dividend is paid on common stock (hence the name *preferred*). If the Inkless Pen Company booms, the holders of common stocks may get larger dividends while the holders of preferred stock get only their fixed dividends. If the I.P.C. is dissolved the preferred stockholders get first claim on whatever is left after the company's debts are settled. Since the earnings on preferred stock are fixed, there is very little fluctuation in its market value.

Preferred stock usually offers good security with moderate dividends. You won't get rich fast from preferred stock!

EXERCISE 2·2

1. Learn the meaning of the words that are italicized in this section. Outline the differences and similarities between those pairs of words in part B.

A	capitalization	retained earnings
	percent return	fixed dividend rate
B	common stock	par value stock
	preferred stock	no-par-value stock
	Class A stock	
	Class B stock	

2·3 How to Read Stock Tables

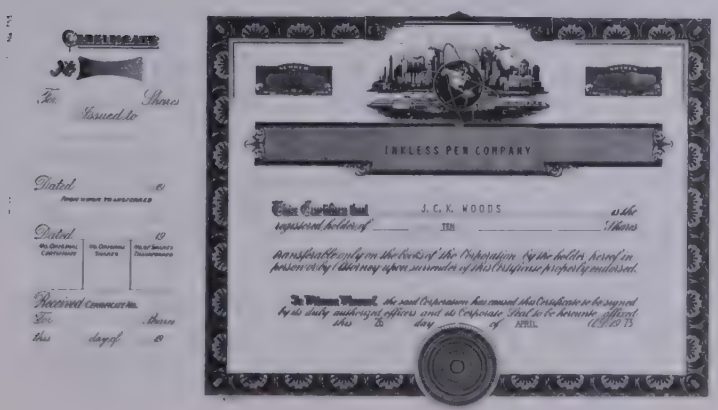
If you are planning to invest in stocks you will want to be able to read the quotations that appear in the daily newspapers. One line from a quotation for the TSE (Toronto Stock Exchange) for December 22, 1971, is shown here.

						1971	
STOCK	SALES	HIGH	LOW	CLOSE	CH'GE	HIGH	LOW
IMP OIL	43274	$32\frac{1}{8}$	$31\frac{3}{8}$	$31\frac{1}{2}$	$-\frac{5}{8}$	$32\frac{3}{8}$	$11\frac{1}{8}$

Reading this line from left to right:

(a) The name of the stock is listed first, in this case, Imperial Oil.

(b) The total number of shares traded during the day was 43,274.



(c) The high price for the day was $\$32\frac{1}{8}$. Notice the use of fractions.

(d) The low price for the day was $\$31\frac{3}{8}$.

(e) The price at closing time (end of day) was $\$31\frac{1}{2}$.

(f) The net change from the previous (business day) closing sale is $-\frac{5}{8}$ (meaning down five-eighths of a dollar).

(g) The high of the 1971 year was $\$32\frac{3}{8}$.

(h) The low of the 1971 year was $\$11\frac{1}{8}$.

EXERCISE 2·3

INDUSTRIALS

TORONTO WEEKLY STOCK QUOTES

Week Ending Dec. 24, 1971

	Sales	High	Low	Close	Ch'ge	High	Low
Abitibi	63223	$7½	7	7⅜	+⅝	8¾	5¼
Abitibi 7½p	210	$49½	49	49½	+1	53½	46½
Acklands	8950	$7¾	7⅜	7¾	+⅜	7¾	5
Acklnd 2p	920	$11¼	11¼	11¼	+⅜	12	8⅛
Acres Ltd	8541	$11⅝	11	11½	+½	13⅝	8⅜
Acres A pr	1496	$41½	40	40	-⅝	43	34½
Acres wt	5596	240	220	230	+15	330	160
A.G.F.M.	7141	$6	5¼	5¼		9	325
Agra Ind	12335	$9⅞	9	9¾	+¾	9⅞	6¼
Alta Gas T	12969	$50½	49⅝	50¼	+¼	53	43½
Alta Gas p	260	$77	75	75		80	71
Alta G D p	212	$140	138	138	-2	147	120
Alta Nat	1561	$21	20	20½	-¼	26	20
Alcan	109916	$20	18½	18⅞	+1	25¼	15⅛
Alcan pr	5070	$27½	26½	27		30	24
Algo Cent	2375	$8¼	7⅞	8¼	+¼	10½	7¼
Algoma St	52216	$13½	13	13¼	+⅜	16¾	11⅛
Algon	100	405	405	405	-20	455	210
Algonq pr	100	400	400	400		405	180
Allarco Dv	2615	$5	450	5	+75	8½	410
Alliance B	2016	360	320	325	-35	415	230
Alumin pr	250	$18⅛	18	18⅛	+⅛	18¼	16½
Almin 2p	192	$36¼	36	36¼	+¼	37	32¼
Anglo-cn	3730	475	450	470	+40	5½	375
Anglo-c pr	1600	$14¼	13⅝	13⅝	-¼	15¼	12⅜
Ang CT 4½	229	$30¾	30	30½	+2⅛	33	28
Ang CT 290	140	$38	38	38		43½	35½
Ang CT 315	205	$42⅝	41	42	+1	46	39
Anthes C p	z30	$80	80	80		83	70
Aquitaine	30568	$25	24⅛	24⅝	+⅛	29¼	20¾
Argus	384	$14	13	13	+1	17¾	10
Args 250 p	150	$32⅛	32	32⅛	+⅛	34½	30½
Args 260 p	60	$35	34¼	34¼	+⅛	35	31
Argus B pr	685	$35¾	34⅞	35¾	+⅞	36	31
Argus C pr	12720	$9¾	9⅜	9½	+½	10¼	8
Asbestos	5441	$26¾	25¼	25½	+¼	37	24¾
Atco Ind	15400	$8⅜	7½	8⅜	+1⅛	10	6½
Atl Sugar	9460	$6½	6⅛	6½	+¼	8⅛	5½
Atl Sug A	765	$15	14⅞	14⅞	-⅛	16½	13¼
Atl Sug pr	545	$63	62⅜	63	-½	66½	59⅛
Atl Sug W	3985	145	130	130	-5	250	115
Auto El	2720	$6	6	6	-¼	7	450
Auto Hard	2400	$7½	7¼	7¼	-⅜	11½	6⅞
Bahama	20600	16	14	14		36	10
Bank Mtl	103775	$18¾	18¼	18½	+¼	18¾	14¼
Bank N S	47545	$31¾	30⅜	30½	-½	32	20⅝
B Cdn Nat	5565	$14¼	13⅞	14	-⅛	14¾	12½
Bartaco	1300	450	450	450	-10	8	410
Bathrst p	100	$7½	7½	7½	+¾	12¾	6
Baton B	53180	$15¼	14¼	15¼		300	14¼
Bay Mills	z20	120	120	120		475	150
Bay Mills p	100	315	315	315	+10	10	300
Beaver Eng	2375	$6⅞	6⅛	6⅛	-⅞	19⅝	5¾
Beaver L	13500	$19⅝	19⅛	19½	+¾	19⅜	13½
Beaver A	740	$19⅜	19⅜	19⅜	+2⅛	19⅜	12¾
Becker B pr	360	$10	9¾	9¾	-¼	10¼	7½
Belding L	z10	$10¾	10¾	10¾		10⅝	7¼
Belding W	1000	280	280	280	-15	365	130
Bell Canad	56281	$47¼	45¾	47⅛	+1¼	50	43¾
Bell A pr	5576	$52½	51½	52½	+¼	55¾	50
Bell B pr	7967	$54¼	52¼	54¼	+½	54¼	50½
Biltmore	z75	$13¼	12	12		13	5
Black B	10255	$5	475	490	+15	5	225
Block Bros	37165	310	295	300		470	250
Bombrdier	3415	$9¾	9¼	9¼	+¼	14⅛	8¼
Bovis Cor	12150	190	175	180	+5	265	115
Bow Valy	28121	$29⅜	27¾	28	-¾	29⅞	15
Bowater	240	475	475	475	+10	475	325
Bow-m pr	57	$32	32	32		38	30
Bramalea	22790	360	320	360	+45	5¼	260
Bramal W	19650	210	185	200	+10	430	140
Brascan	60557	$18⅝	18⅛	18½	+½	20½	15
Brdg Tank	600	450	450	450		5¼	220
Bright	6085	$16	15½	16		16¾	15
Brinco	16143	$5⅞	5½	5⅞	+¼	6¼	480

1. Study the quotations shown:
   (a) What stock exchange is reporting?
   (b) For what days is this report?
   (c) How many Allarco shares were traded?
   (d) What was the weekly high for Acres Ltd.?
   (e) What was the weekly low for Acres Ltd.?
   (f) What was the closing for Acklands?
   (g) What was the change for Acklands?
   (h) What was the year's high for Algo Cent?
   (i) What was the year's low for Algo Cent?
   (j) Look at Anthes. What does the Z30 mean? (Any order under 100 shares is called an *odd lot*.)
   (k) Notice there are four different Atlantic Sugar. *Atl Sug A* indicates Class A stock, while *Atl Sug pr* indicates preferred stock. Name some other stocks listed as Class A; as preferred stock.
   (l) Explain the *highs*, *lows*, *closing* and *change* for Allarco.
   (m) Explain the *highs*, *lows*, *closing* and *change* for Atco.
   (n) Which, Bartaco or Bank N S, showed the most loss?
   (o) What stock of those shown, showed the most gain over the week? the most loss?

2. Different newspapers use variations of the listing in question 1.

Vancouver

VANCOUVER CLOSING STOCKS

Vancouver Stock Exchange--Dec. 22

Complete tabulation of Wednesday transactions. Quotations in cents unless marked $. Net change is from previous board-lot closing sale.

Stock	Sales	High	Low	Close	Net Ch'ge
Mines					
Adera	22500	33	32	32½	-½
Afton	314700	134	108	121	+9
Ajax Ltd	3000	11	11	11	
Alwin	3400	65	62	64	-1
Anuk	15500	25	24	25	
Ardo	500	16	16	16	
Arctic	2500	9	8½	9	+2
Atlas	2000	34	32	32	
Bath Nors	9150	85	84	85	
Beaumnt	8000	9	9	9	+1
Blue Glch	7500	48	48	48	

Industrials					
Block Bros	1600	305	300	300	
BC Forest	100	$20½	20½	20½	
BC Sug	200	$19⅜	19	19	
Burrad	700	$7¼	7	7	
CP Inv pr	100	$24⅞	24⅝	24⅜	
Cdn Pac	800	$13⅞	13¾	13⅞	-⅛
Canterra	2700	105	95	100	
Capt Int	8150	$6⅛	5¾	6	+¼
Cominco	100	$22⅜	22⅜	22⅝	
Comptrex	2000	38	37	38	
Cornat	1811	165	160	165	+5
Dawson	200	650	650	650	+50
Dogs Suds	500	25	25	25	
Doman pr	40	$39	39	39	+1
Doman	2275	$9⅞	9¾	9¾	-⅛
EDP Ind	300	35	35	35	
Field	1215	$13¾	13½	13½	
Finn Tr	1000	$12½	12¼	12½	+⅛
First City	400	$7¾	7¾	7¾	
Gt Nat Ld	1450	110	105	105	-5
Grouse	400	245	240	240	-10

Calgary

CALGARY CLOSING STOCKS

Calgary Stock Exchange--Dec. 22

Complete tabulation of Wednesday transactions. Quotations in cents unless marked $. Net change is from previous board-lot closing sale.

Stock	Sales	High	Low	Close	Net Ch'ge
Industrials					
Edm Speed	2000	65	60	65	+15
KSF Chem	200	180	180	180	-10
Parkland Bf	200	30	30	30	
Oils					
Amalta	2000	12½	12½	12½	+½
Amer Eagle	7600	80	75	75	-5
Barons	2000	5	5	5	+1

Bashaw Led	2500	4½	4½	4½	
Inqua	7000	60	50	50	-9
Jagor	1000	20	20	20	-1
Monterey A	1100	21	21	21	+1
Nitracell	500	30	30	30	-1
North Cont	5000	2	2	2	-1
Petromine	1000	26	26	26	
Mines					
Alta Copper	2000	40	40	40	
Bird River	4000	65	55	65	+10
Bright Star	18700	110	110	110	+3
Contact	4000	55	50	55	-5
Cultus	1000	20	20	20	
Terra	400	240	230	240	
Trans Yuk	2000	5½	5½	5½	

Total sales: 64,250

(a) What two stock exchange listings are shown?
(b) Three quotations are underlined for each exchange. Explain all the information given for each.

(c) In the Calgary Stock Exchange note that for *Parkland Bf* there is no quotation for Net Change. This is the case when there has been insufficient trading to show a trend. For what other stock was there insufficient trading to establish a trend at CSE?

(d) What was the high price for Barons? Trans York? Beaumont? Oil and mining stocks that sell for less than $1 per share are known as *Penny Stocks*. These usually sell in units of 500. Industrial stocks selling for less than $25 sell in units of 100; between $25-$100 in units of 25; if over $100, in units of 10. The units that stocks sell in are called *Board Lots*. Many stockbrokers will not deal in penny stocks.

2·4 Evaluating Stocks

There are several comparisons investors make when attempting to determine whether or not a particular stock is worthwhile as an investment. These comparisons involve market value of the stock and dividend paid. In Section 2·2 reference was made to percent return. The common term for this is *Yield Rate* or *Dividend Yield*

$$\text{Dividend Yield} = \frac{\text{annual dividend}}{\text{market value of share}} \times 100\%$$

Example 1

In 1971 Hollinger Mines paid $1.60 dividends on stock with a market value of $36.50. The same year O'Brien paid 1¢ dividends on stock that had a market value of 14¢ a share. Compare the two Dividend Yields.

Solution

$$\text{Dividend Yield}_{HM} = \frac{\text{annual dividend}}{\text{market value of share}} \times 100\%$$

$$= \frac{1.60}{36.50} \times 100\%$$

$$= 4.38\%$$

$$\text{Dividend Yield}_{O'B} = \frac{1}{14} \times 100\%$$

$$= 7.14\%$$

Hence we see that O'Brien had a *better* Dividend Yield than did Hollinger Mines. This means that had a person invested $1000 in each company, he would have collected more in dividends from O'Brien than from Hollinger Mines, even though the dividend per share was higher for Hollinger Mines.

Current quotations are constantly being flashed on a Quotation Screen in the broker's board room.

A second ratio used is the *price-earnings ratio* (P/E Ratio). This ratio is expressed in the form *n*:1 but normally in the newspaper listing only the *n* is given for it is expected the reader will interpret this accordingly.

$$\text{P/E Ratio} = \frac{\text{current price per share}}{\text{last 12 months earnings per share}}$$

Example 2

Compare the P/E Ratio and interpret the results for Rio Algom when:

1970 annual dividend $.40 market value $12.00

1971 annual dividend $.40 market value $15.00

Solution

$$\text{P/E Ratio} = \frac{\text{current price per share}}{\text{last 12 months earnings per share}}$$

$$\text{P/E Ratio}_{1970} = \frac{12}{.40}$$

$$= 300 \text{ or } 300{:}1$$

$$\text{P/E Ratio}_{1971} = \frac{15}{.40}$$

$$= 375 \text{ or } 375{:}1$$

In 1970 the stock sold at 300 times the annual earnings per share. In 1971 the stock sold for 375 times the annual earnings. This indicates that these stocks are in greater demand, hence investors are showing confidence in this company. One must also compare this change with similar ratios for other companies in the same line. If the other ratios also went up this would indicate Rio Algom followed the trend. A drop in the P/E Ratio is a warning signal that suggests the investors should study what is happening to the company.

Other ratios are used as indicators of the standings of companies. Remember these indicators, like the P/E Ratio are relative—they must be viewed in light of what these same ratios are doing for other stocks as well as the one being investigated. If you become interested and active in trading stocks you may wish to investigate *Dividend-Payout Ratio*, *Gross Profit Margin*, and *Net Profit Margin*.

EXERCISE 2·4

	MINING COMPANY DIVIDENDS			
	CLOSING		DIVIDENDS	
	1970	1971	1970	1971
Bethlehem	$14.00	$17.75	$ .65	$ .60
Falconbridge	90.00	80.00	3.50	2.75
Grandisle	6.00	7.60	1.20	.98
Liberian	14.30	10.25	1.25	2.22
Upper Canada	2.40	1.66	.05	.02
Nor Nan	.39	.56	.006	.012
Denison	.32	.23	1.40	1.40
Keer Addison	7.50	7.25	.56	.45
Pine Point	35.00	24.25	7.00	4.50

1. On the basis of the information given above find the P/E Ratio for each company for 1970 and 1971. Also find the Dividend Yield. In your opinion which company would be the best investment? Base your decision on the concepts covered and explain your choice.

2. *The Financial Post* assists its readers with the introduction shown on opposite page. Notice it gives the *Ind. Yield* (Dividend Yield) and the P/E Ratio. The dividend per share is given by *Earnings per Share*. The second last column is the latest and is to be used unless it represents only part of a year in which case the last column is used.

How to read FP's tables

To make it easier for readers to relate earnings figures to the stock quotation, FP has developed a system of presenting price, dividend and earnings data. Here are some examples of earnings figures and how to interpret them:

Earnings per share

	1970 or latest interim		1969 or latest fiscal year	
1.	'70 Dec	$ .81	'69 Dec	$1.21
2.	'70 March	1.12	'69 March	1.02
3.	'70 May 8m	.53	'69 Sept	1.00
4.	Sept 9m	2.05	'69 Dec	3.61
5.	June 26w	1.10	'69 Dec	2.25

Example 1. Earnings per share in year ended Dec./70 were 81¢ a share; in year ended Dec./69, $1.21.

Example 2. Earnings per share in year ended Mar./70 were $1.12 a share; in year to Mar./69, $.02

LETTER CODE

Small Capital Letters:

A After participation
B Before participation
D Deficit
E Exact number of shares
F Maximum participation
G Less than board lot
L Date of transaction
P Paid or payable in last 12 mos. to date
R Dividend in arrears; or dividend omitted
x Times dividend earned
U Payable in U.S. funds
* New figure

Exchange on which largest volume took place in the week:

C Calgary Stock Exchange
M Montreal or Canadian Stock Exchange
T Toronto Stock Exchange
V Vancouver Stock Exchange

Where there have been no transactions during the week covered the word "nil" is shown under sales.

P/E Ratio - Price/Earnings Ratio

Example 3. Earnings per share in abbreviated eight-month fiscal period ended May/70 were 53¢ a share; in year ended Sept./69, $1.00 per share.

Example 4. Earnings for nine months ended Sept./70 were $2.05 a share; for year ended Dec./69, $3.61.

Example 5. Earnings for 26 weeks ended June/70 were $1.10 a share; for year ended Dec./69, $2.25.

When there are no interim figures, the figures are for the two latest fiscal years. When there are interim figures, latest interim earnings figures are shown in the left hand margin, and the latest fiscal year in the right hand column.

Twelve mos. cumulative earnings, used to calculate the P/E ratios, for those cos. which report in U.S. funds have been converted to Cdn. Dollars.

Week Ended Dec. 23

Range 1971 High $	Range 1971 Low $	Company	Indic Div. Rate $	High $	Low $	Close or Latest $	Net Chge.	Sales 100s	P/E ratio Ind. Yld. %	P/E ratio latest 12 Mos.	Earnings per Share 1970 or latest interim	$	Earnings per Share 1969 or latest fiscal year	$
14.00	8.75	Finlayson Ent. A	0.90	LDec.10,1971		14.00	....	nil	6.4	..	'70 Dec	x8.41	'69 Dec	x5.11
...	...	Do. B	0.30	LJul.11,1969		17.00		nil	1.8	27.9	Jun 6m	.77	'70 Dec	1.67
14.75	9.50	Finning Tractor & Eq. .	0.30	12.75	12.00	12.50	+0.75	21V	2.4	8.1	Sep 9m	1.26	'70 Dec	1.23
9.25	6.50	First City Fin. Corp. .	P0.12	7.75	7.50	7.75	+0.25	12V	1.5	..	Sep 9m	.70	.....	...
9.75	8.50	Fiscal Inv.	0.10	LDec.10,1971		9.00	....	nil	1.1	15.3	'69 Apr	.59	'68 Apr	.30
0.20	0.14	Do. pfd.	0.01	0.20	0.20	0.20	unch.	77M	5.0	..	'69 Apr	x2.16	'68 Apr	x3.29
17.75	13.00	Fittings	0.60	LDec. 3,1971		15.00	....	nil	4.0	11.2	Sep 9m	.98	'70 Dec	1.29
1.40	0.75	Fleet Mfg.	...	0.88	0.84	0.88	+0.03	38T	..	..	Sep 3m	.01	'71 Jun	.09
10.50	6.00	Fleetwood Corp.	...	8.50	8.00	8.00	+0.62	11M	..	..	Sep 9m	D.68	'70 Dec	D1.87
72.50	55.50	Ford Motor Co.	U2.60	71.87	70.00	71.87	+4.87	7T	3.6	12.7	Sep 9m	U4.24	'70 Dec	U4.77
100.00	65.00	Ford of Canada	3.00	85.00	81.00	85.00	+4.00	43T	3.5	9.2	Sep 9m	6.75	'70 Dec	8.37
1.50	0.40	Founders Group	...	LDec.10,1971		0.45	....	nil	..	..	.....	....	.....	...
0.65	0.15	4-F Foods Ltd.	...	0.19	0.15	0.15	unch.	25V	..	..	Nov 6m	.05	.....	...
14.50	8.50	Four Seasons	...	14.50	14.00	14.50	+0.37	59T	..	18.1	Sep 9m	.59	'70 Dec	.66
6.50	4.00	Do. warr.	...	6.50	6.00	6.50	+0.50	20T	..	..	....	....	.....	...
21.00	15.25	FPE-Pioneer Elec.	0.56	18.00	17.50	17.50	+0.50	6T	3.2	7.7	Sep 3m	A.31	'71 Jun	A2.04
83.00	60.00	Do. 5½% cv. pfd.	2.75	LDec.17,1971		69.00	....	nil	4.0	..	'71 Jun	x16.33	'70 Jun	x11.82
21.00	11.25	Fraser Co.	P0.15	12.37	12.00	12.37	+1.00	48M	1.2	..	Oct 40W	D1.87	'71 Jan 52W	4.04
6.12	4.10	Freiman	0.20	6.00	5.87	6.00	unch.	4T	3.3	15.0	Oct 9m	.21	'71 Jan	.38
18.50	14.00	Fruehauf Tr.	0.75	16.50	16.50	16.50	unch.	6T	4.5	7.0	Sep 9m	1.87	'70 Dec	1.90
4.50	3.25	Fulcrum Inv.	...	3.65	3.65	3.65	+0.25	2M	..	..	....	....	.....	...
9.12	7.50	Do. pfd.	P0.65	LDec.17,1971		7.50	....	nil	8.7	10.3	'70 Dec	.73	'69 Dec	.64

Courtesy of The Financial Post

(a) What is the P/E Ratio for
  (i) Fiscal Inv.
  (ii) Ford of Canada
  (iii) Fraser Company

(b) What is the Dividend Yield (Ind. Yield) for
  (i) Finning Tractor & Eq.
  (ii) Fittings
  (iii) Fruehauf Tr.

(c) From the information shown, what stock would appeal to you? Why?

(d) Check the P/E Yield for:
  (i) Fiscal Inv.
  (ii) Fulcrum In. pfd.

3. Calculate the Dividend Yield and the P/E Ratio for each, given the additional data in each case.

   (a) Closing price of Canadian Jamieson Mines Limited for 1971 was $1.23.

Canadian Jamieson Mines Limited

DIVIDEND NO. 6

Notice is hereby given that a dividend of twelve and a half cents (12½¢) per share has been declared, payable on January 24, 1972 to shareholders of record at the close of business on January 10, 1972.

By Order of the Board:

R. C. BRAGAGNOLO,
Secretary.

The transfer agents of the company are: Guaranty Trust Co. of Canada, 88 University Ave., Toronto, Ontario, who should be notified promptly of any change of address, or change of share ownership.

Dated at Timmins, December 9, 1971.

   (b) The dividend for Cassiar Asbestos Corp. Ltd., was paid four times in 1971. The closing price in 1971 was $19.50.

Cassiar Asbestos Corporation Limited

Notice is hereby given that an interim dividend of fifteen cents (15¢) per share plus an extra dividend of five cents (5¢) per share has been declared payable in Canadian Funds on January 28, 1972 to shareholders of record at the close of business on December 31, 1971.

By Order of the Board,

J. R. SCRIMGER, F.C.I.S.,
Assistant Secretary.

   (c) Conwest Exploration Company Ltd. paid the dividend twice in 1971. The closing price in 1971 was $7.50.

Conwest Exploration Company Limited

Notice is hereby given that an interim dividend of six cents (6¢) per share has been declared payable in Canadian Funds on January 31, 1972 to shareholders of record at the close of business on December 31, 1971.

By Order of the Board,

J. K. SCRIMGER,
Secretary.

Toronto, Ontario,
December 7, 1971

VANCOUVER STOCK EXCHANGE

prices for the week ended Dec. 14

1971 High	1971 Low		Shares Issued	High	Low	Close	Change	Volume
59	27	Adera	2,663	41	29	33½	+ 4½	386,700
167	10	Afton Mns	2,424	132	92	115	-15	1,976,700
18	9	Ajax Ltd	2,207	10	10	10	...	500
38	19	Alice Lk	1,935	24	22	22	- 2	2,500
93	55	Alwin	3,125	66	62	62	- 4	16,260
46	18	Anuk	5,200	29	18½	29	+ 3	161,550
61	11	Ardo Mns	2,455	19	17	17	- 1½	21,500
35	16	Arlington	3,899	18	17	17	...	2,500
22	6	Arctic	3,507	8	6	7½	...	8,800
77	20	Atlas	4,869	35	28	32	+ 2	6,450
100	30	Avino	3,622	32	30	30	...	2,000
450	61	Ballinderry	1,721	105	98	104	+ 5	26,466
305	51	Bthrst Nrse	4,176	86	80	83	- 4	33,775
30	12	Bear Intl	2,263	15	12	15	...	4,000
24	4	Beaumont	2,330	7	6	7	+ 1	10,000
30	12½	Bell Moly	1,974	12½	12½	12½	...	2,500
49	10	Benson	2,696	15	12	14	+ 3	6,000
1150	900	Bison	3,740	975	930	930	-10	1,903
118	25	Black Giant	3,000	25	25	25	...	3,000
54	20	Blue Gulch	1,433	47½	45	47	+ 1	17,000
11	4	Blue Star	3,575	7	6	7	+ 1	5,000
57	20	Boswell	2,582	23	20	20	...	4,000
49	8	Boundary	1,917	10	9	9	...	7,500
50	29	BrenMac	6,825	37	36	36	- 2	48,000
94	9	Brycon Ind	1,873	15	13	13	- 3	5,000
19	8	Buchanan	2,612	17	15	15	...	55,000
76	28	Bullion Mtn	2,369	70	44	68	+ 5	524,700
50	20	Buval Exec	5,000	36	31	34	- 1	47,700
19	10	Calico	2,486	19	18	18	+ 2	19,500
154	45	Calta	3,193	84	66	80	+12	155,432
56	25	Cambridge	2,155	38	32	34	...	22,000
69	20	C Barranca	2,475	23	21	22	- 3	3,500
38	14	Canol	1,851	18	17	18	...	7,000
64	18	Car Bell	2,066	24	23	24	...	2,500
125	70	Car Gold	2,760	100	90	90	-10	3,800
59	18	Casc Moly	2,115	19	19	19	...	500
16	5	Cass Cons	4,300	10	9	10	...	5,777
125	35	Casino	2,826	58	52	55	- 2	10,600
31	12	Chataway	3,716	18	17	17	...	9,500
196	30	Churchill	3,041	66	56	66	1	20,200
500	300	Coast Cop	3,012	300	300	300	...	100

4. Some financial sheets expect the reader to know whether or not a listing is in dollars or cents. The mining shares shown are listed in cents only. Interpret the information for each listing underlined. What is the common name for stocks that sell for less than a dollar per share?

5. Calculate the Dividend Yield for each for 1971 using the *Indicated Dividends* for 1971 and the *Closing Prices* for the current week. Compare your answers with the given yield in the listings.

(a) Abitibi pfd. (b) Acklands pfd.
(c) Acres "A" pfd. (d) Alta N Gas
(e) Alum C pfd. (f) Algoma Cen

6. Calculate the *P/E Ratio* for each for 1971 using the *Closing Prices* for the current week and the *Dividends Indicated* for 1971. Compare your ratios with those given in the listings.

(a) Agra Ind. (b) Alta Gas Tr.
(c) Algoma Stl. (d) Algoma Cen.
(e) Alcan Alum. (f) Ang C Pulp

Toronto, Montreal & Canadian

Week to Friday, December 10, 1971

Industrials

(d) Deficit earnings * Plus stock x Odd lots

Yields are based on 1971 indicated rates (including extras) and closing price Friday. Price earnings ratio is based on latest fiscal year and closing price Friday. Where no trading in week Thursday's bid is shown under column Current Week--High -- and asked under column--Close. In the earnings per share column, figure given under preferred earnings is times preferred dividends earned.

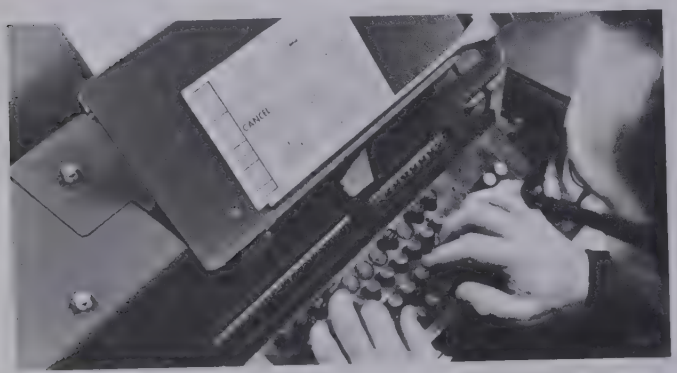
Yield %	Price/ Earn. ratio	1971 High	1971 Low	Stocks:	Dividends Paid 1970	Dividends Indic. 1971	Earnings 1970 or latest fiscal year	Earnings Latest Interim	Current Week High	Current Week Low	Current Week Close	Current Week Net Chg.	Sales 100s
..	26.0	8¾	5¼	Abitibi P.	.32	..	Dec .24	Sept9m .12	6⅝	6¼	6⅜	+ ⅛	444
8.0	...	53¼	46½	" pfd	3.75	3.75	Dec 6.07	.....	47½	46½	47½	+ 1	3
..	...	7¾	5	Acklands	.24	..	Nov 1.53d	Aug9m .19	7¼	6⅝	6⅞	+ ⅛	72
8.3	...	18½	14	" pfd	1.50	1.50	.....	.....	16	16	16	- 2½	1
..	...	12	8⅛	" 2nd pfd	.96	..	.....	.....	11	10⅞	11	.....	2
0.4	...	13⅝	8½	Acres	..	.05	Decl4m .54	Sept9m .73	11¼	10¾	10⅞	- ⅝	67
8.5	...	43	34½	" 'A' pfd	1.95	3.60	Decl4m 1.90	.....	42½	41¼	41⅜	- 1⅛	18
..	...	3.30	1.60	" wts	...	..	.....	.....	2.40	2.05	2.05	- .35	42
..	...	19	11⅛	Admiral	..	..	Dec 3.08d	Sept9m .40	15	..	16	.....	..
4.5	16.2	9	3¼	AGF Mngt	.30	.19	Nov .26	Aug9m .15	4.25	4.20	4.20	+ .05	16
0.7	10.8	9¾	6¼	Agra Ind.	..	.06	71July .83	.....	9	8½	8¾	- ¼	84
..	36.9	6	1.95	Ahed Music	..	..	Nov .13	Aug9m .25	4¾	4¾	4¾	- .05	20
..	11.2	17¾	11⅜	Aimco	..	..	July 1.40	Apr9m .77	15½	15½	15½	- ¼	10
3.1	20.3	53	43½	AltaGasTr	1.00	1.50	Dec 2.35	Sept9m 1.84	49	47⅝	48¾	+ 1	70
6.3	...	80	61	" 4¾ pfd	4.75	4.75	Dec 4.80	.....	75	75	75	.....	1
4.1	...	147	120	" D pfd	5.37½	5.37½	Dec 4.80	.....	135	135	135	+ 4	1
5.0	13.9	26	20	Alta N Gas	1.00	1.00	Dec 1.44	Sept9m 1.01	21	21	21	+ 1	4
4.8	10.2	10½	5⅝	Algoma Cen	.38	.40	Dec .81	Sept9m .55	8¼	7¾	8	- ¼	42
4.0	6.2	16¾	11⅛	Algoma Stl	.50	.50	Dec 2.03	Sept9m .79	13	12⅛	12½	....	292
..	56.9	4.55	2.10	Algonquin	..	..	Dec .08	Sept9m .15	4.40	4.40	4.40	- .15	10
..	...	4.05	1.80	" pfd	..	..	Dec .79	.....	4	4	4	....	9
5.6	8.5	25¼	15⅞	AlcanAlum	1.20	1.00	Dec 2.08	Sept9m 1.37	18½	17¼	17⅞	+ ⅛	1155
6.7	...	30	24	" 4¼ pfd	1.70	1.70	Dec 29.04	Mar3m 22.75	26¼	25	26¼	+ 1	63
..	5.6	8½	4.10	Allarco Dev	..	..	Oct .73	Mar6m .12d	4.20	4.15	4.15	+ .05	4
5.5	...	18¼	15	Alum C pfd	1.00	1.00	Dec 26.36	.....	18	18	18	.....	1x
6.3	...	37	32¼	" 2nd pfd	2.25	2.25	Dec 26.36	.....	36¼	36	36	.....	19
..	12.2	4.15	2.30	Alliance	..	..	Dec .23	Sept9m .11	3.05	3.00	3.05	+ .25	49
1.2	...	8	5½	" pfd	.70	.70	Dec 5.39	.....	7	...	9	.....	..
..	...	2.45	.90	Ambassador	..	..	71Mar .03d	.....	1.35	.90	1.30	- .05	395
..	...	...	...	Amerada H	.30*	.30	Dec 3.22	Sept9m 2.95	...	...	...	.....	..
..	...	...	...	" pfd	3.50	3.50	Dec 3.13	.....	...	...	...	.....	..
..	22.1	2.05	.60	Am-Metro	..	..	Dec .07	June6m .02	1.40	1.15	1.15	- .40	8
8.6	8.3	5½	3¾	Ang C Pulp	.36	.36	Dec .50	Sept9m .20	4.20	4.00	4.00	- .20	4
8.8	...	15¼	12⅜	" pfd	1.12½	1.12½	Dec 6.95	.....	13½	...	14	.....	..
7.5	...	33	28	AngCT 4½	2.25	2.25	Dec 6.09	.....	30	29¾	29¾	- ¼	1
7.7	...	38¼	33	" 2.65	2.65	2.65	Dec 6.09	.....	34	34	34	.....	1
7.5	...	43	35½	" 2.90	2.90	2.90	Dec 6.09	.....	39	37½	37½	- ¾	1
7.7	...	46	36¾	" 3.15	3.15	3.15	Dec 6.09	.....	41	41	41	.....	1
..	...	16½	11	Ang Pm H	..	..	.....	.....	11⅞	...	13½	.....	..
6.8	...	83	70	AnthesB pf	5.50	5.50	71Mar 5.34	.....	80	80	80	.....	1
6.3	...	83	70	AnthesC pf	5.25	5.25	71Mar 5.34	.....	80	80	80	.....	1x
..	35.0	8½	2	Aquablast	..	..	June .10	.....	3.70	3.30	3.65	+ .15	68

2·5 So You Want to Trade in Stocks

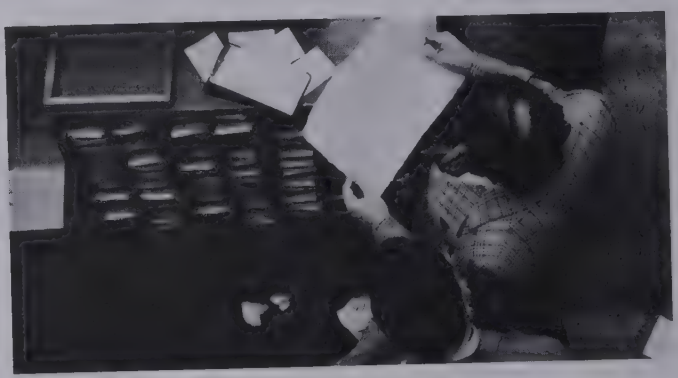
The Story of a Transaction on The Toronto Stock Exchange

Mrs. Jones in Toronto phones her account executive to tell him that she would like to sell her 100 shares of Typical Manufacturing Company stock.

Meanwhile, you telephone your account executive and tell him to buy 100 shares of Typical for you.



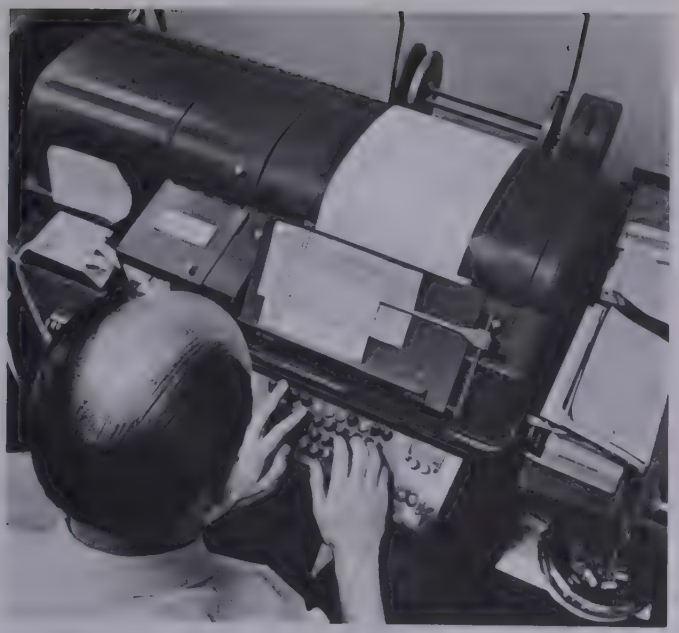
Your order is teletyped from the Merrill Lynch office where you do business...



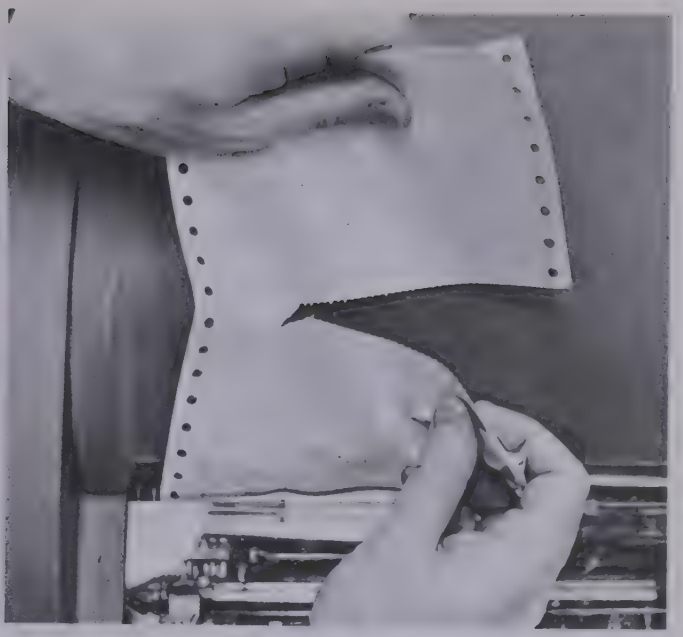
to the Wire Room in Merrill Lynch's Toronto headquarters and simultaneously directly to the floor of the Toronto Stock Exchange...

At the Exchange, it is given to a Merrill Lynch floor broker...

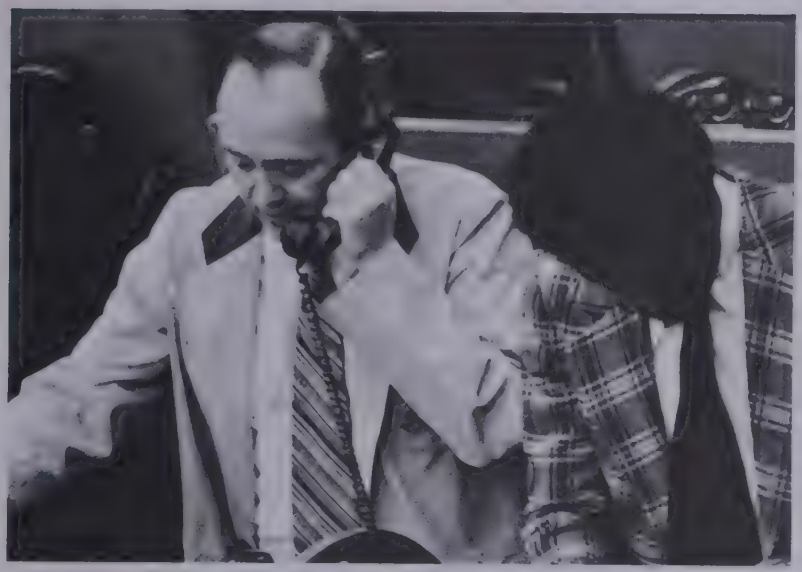
who takes it to the appropriate trading post and bargains for the best possible price for you. The floor broker representing Mrs. Jones is there, too.



When a price has been agreed on, a confirmation is teletyped directly to the originating office and also reported back to the Wire Room.



In the office, the wire clerk receives the confirmation and relays it to your account executive...



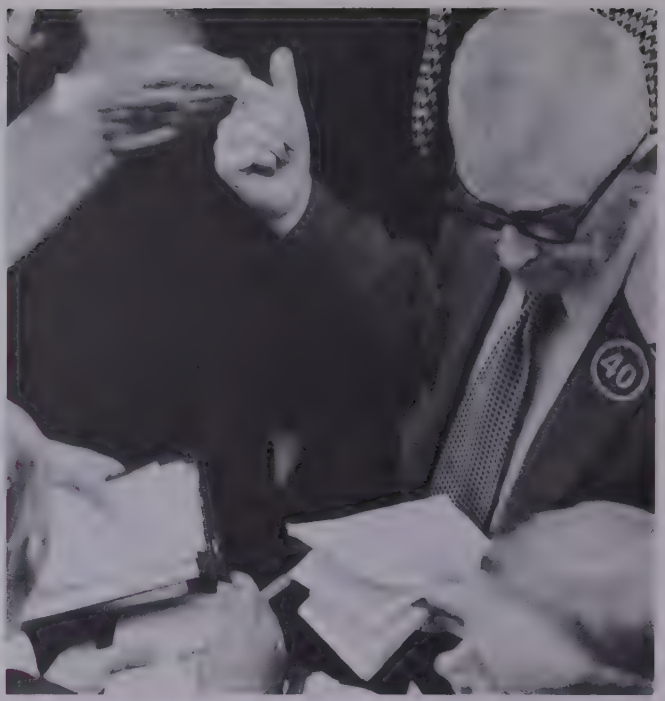
who telephones you to tell you the price at which 100 shares of Typical have been bought for you. And Mrs. Jones's account executive notifies her that her stock has been sold.

You have bought your stock and Mrs. Jones has sold hers in a matter of two or three minutes without ever meeting, thanks to the marketplace that is the Toronto Stock Exchange.

EXERCISE 2·5

1. Organize a trip to a broker's office and see how business in such an office operates. Plan questions you will ask your host in the office.
2. Ask a representative of a broker's house to visit your class to discuss the stock market business. Include in your letter the topics you would like him to discuss. Prepare a set of questions you will ask when he visits your class.
3. Collect from newspapers clippings relating to the stock market that are of interest. Be prepared to explain what affect each story might have on the stock market.

2·6 Let's Buy Stock



Projects

Project 1

Purpose

To simulate actual purchases of stock and see the subsequent results.

Method

1. Obtain and study for a period of time, financial and business pages of daily newspapers and financial newspapers.
2. Plan how you would buy $1000 worth of shares. Keep a weekly record over a period of two or three months (or longer) of what happens to your stocks. A chart like this will help you.

	NAME OF STOCK	NUMBER PURCHASED OR SOLD	PURCHASE PRICE	WEEKLY CLOSING PRICE: FIRST WEEK	SECOND WEEK	THIRD WEEK	...	...	NET CHANGE	PROFIT OR LOSS	BROKERAGE AND TRANSFER FEES
1	BBE	200	1.32	1.35	1.32	1.49			+.17		5.00
2	CDE	4600	1.79	1.79	1.78	1.78	1.80	1.89			7.20
3	GHJ	300	1.01				1.03	1.05			2.50
									GROSS PROFIT OR LOSS		
									NET PROFIT OR LOSS		

You may decide after several weeks to sell a particular stock and purchase some other stock. This is illustrated in the chart.

You will want to read thoroughly the business and finance pages in order to get hints of stocks that may grow rapidly or to watch for hints that might indicate some stocks will be falling. What type of international or national news items would cause stocks to rise? to fall?

Do You Have A Computer?

Programs can be built or obtained that simulate and speed up stock market fluctuations. These programs can produce daily news items some of which may influence the market quotations. The object is for the students to identify those news items which may affect market quotations and to identify the nature of the change, and which stocks will be affected.

Project 2

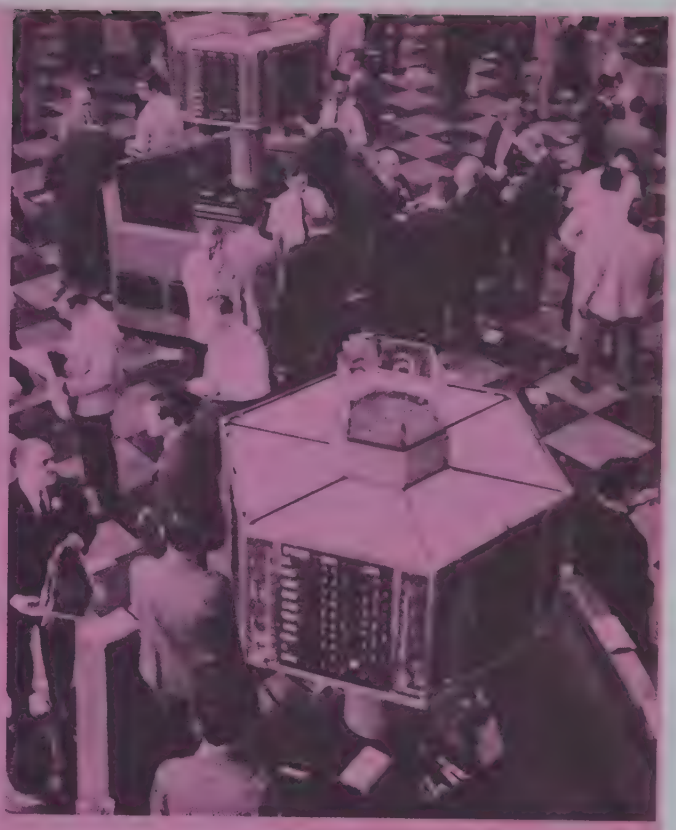
Obtain permission of your school administration before conducting this project.

Purpose

To obtain actual experiences in the stock market.

Method

1. Form an Investment Syndicate (club) in your school.
2. Each student may purchase one $1.00 share in the Syndicate.
3. Each month the club meets to discuss what buying and selling (if any) will be done.
4. A set of "directors" is elected and rotates on a regular basis (usually the rotation is every three months).
5. The directors make the final decision as to what stock is bought and sold.
6. An adult must do the actual business with the brokerage house.
7. At the end of the school year the Syndicate may dissolve by selling all stocks and sharing the profits or losses.



Trading on the Toronto (above) and Montreal Stock Exchanges.

What Is a Good Investment?

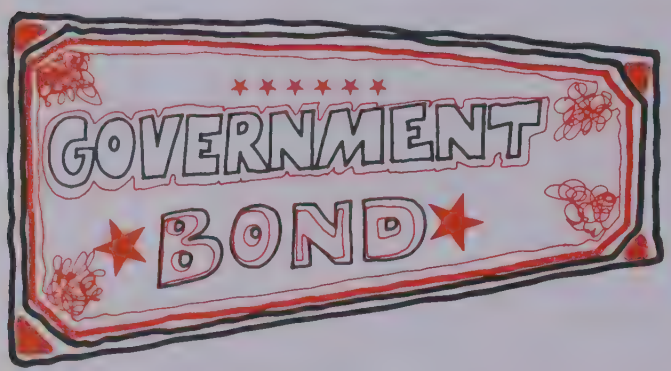
You have probably heard about persons who have invested small amounts in stocks and realized a fortune. Success stories are always interesting and tantalizing. The wise investor will always be extremely cautious about any "get-rich-quick" scheme. The following is a guide to wise investments.

Is Your Principal Secure?

Are you reasonably certain you will get back at least as much as you invested?

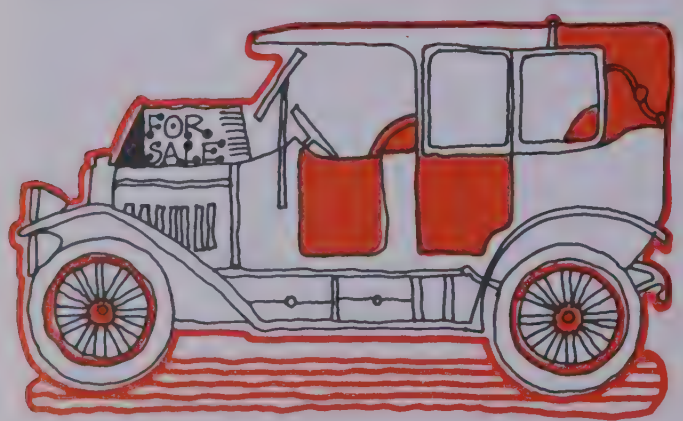
Is It Investment or Speculation?

An investment has a nominal rate of interest with a minimum risk. A speculation indicates a possible high rate of interest with a greater risk.



Can You Sell Your Investment?

Can you sell at any time you wish?



Will The Investment Yield an Income?

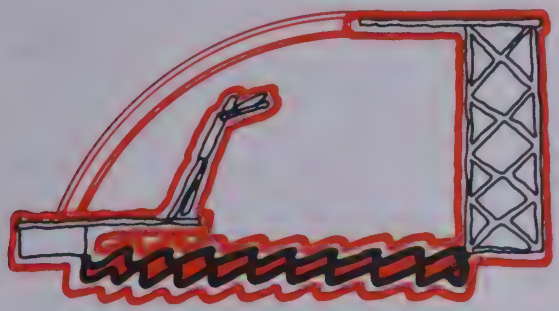
Is your investment likely to yield an income even in a slump?

How Often Must You Reinvest?

It is time consuming to find good investments. Long-term stocks or bonds save you the trouble of reinvesting.

Are You Investing Modestly at First?

You can learn as well with a $100 investment as you can with one of a $1000.



Are You Investing Only Surplus?

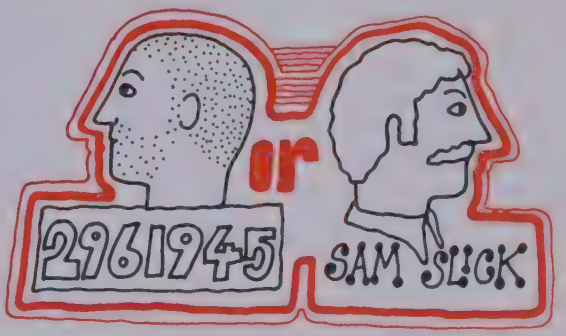
Only invest that which is in excess of your savings. Do not invest the money needed for daily living in the hopes you will get it back before you need it.

Have You Invested In Several Kinds of Securities?

If any one area of the economy or stock market takes a slump, a complete disaster is not experienced.

Have You Investigated the Reliability of the Company?

Study the company's prospectus and consult your broker about persons who manage the business in which you contemplate investment.



Are You Acting on a Friendly Tip?

Only act on advice from a Financial advisor. If the tip is so good how come your friend isn't making money?

2·7 What Does It Cost to Buy and Sell Stocks?

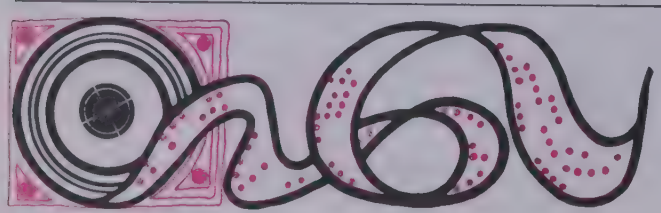
Your agent will charge a *brokerage fee* (commission) according to a standard scale of rates. These rates vary according to the amount of money involved in the transaction. Currently, the average is under 2%. In addition, some provincial governments levy a *transfer tax* (similar to a sales tax) on all stock sales. There is no tax on purchases. The following indicate the types of tables used.

SHARES SELLING AT $1.00 AND BELOW

Price	Commission Rate Per Share	Per Thousand
Under $.05	$.002	$ 2.00
At $.05 and under $.10 ..	$.0035	$ 3.50
At $.10 and under $.20 ..	$.005	$ 5.00
At $.20 and under $.30 ..	$.0065	$ 6.50
At $.30 and under $.40 ..	$.008	$ 8.00
At $.40 and under $.50 ..	$.009	$ 9.00
At $.50 and under $.65 ..	$.011	$11.00
At $.65 and under $.75 ..	$.014	$14.00
At $.75 and under $.85 ..	$.016	$16.00
At $.85 and up to and including $1.00	$.0175	$17.50

SHARES SELLING ABOVE $1.00
(based on 100 shares)

Amount of Transaction	Commission Rate
Above $100 to $800	$2\frac{3}{4}\%$ less $ 1.00
Above $800 to $1,100	2% plus $ 5.00
Above $1,100 to $1,500 ..	1% plus $16.00
Above $1,500 to $3,000 ..	$\frac{1}{2}\%$ plus $23.50
Above $3,000 to $6,500 ..	$\frac{1}{4}\%$ plus $31.00
Above $6,500	$\frac{1}{6}\%$ plus $36.42



100 SHARES - $1.00 to $2.99

PRICE	EVEN	1¢	2¢	3¢	4¢	5¢	6¢	7¢	8¢	9¢
$1.00	1.75	1.78	1.81	1.83	1.86	1.89	1.92	1.94	1.97	2.00
1.10	2.03	2.05	2.08	2.11	2.14	2.16	2.19	2.22	2.25	2.27
1.20	2.30	2.33	2.36	2.38	2.41	2.44	2.47	2.49	2.52	2.55
1.30	2.58	2.60	2.63	2.66	2.69	2.71	2.74	2.77	2.80	2.82
1.40	2.85	2.88	2.91	2.93	2.96	2.99	3.02	3.04	3.07	3.10
1.50	3.13	3.15	3.18	3.21	3.24	3.26	3.29	3.32	3.35	3.37
1.60	3.40	3.43	3.46	3.48	3.51	3.54	3.57	3.59	3.62	3.65
1.70	3.68	3.70	3.73	3.76	3.79	3.81	3.84	3.87	3.90	3.92
1.80	3.95	3.98	4.01	4.03	4.06	4.09	4.12	4.14	4.17	4.20
1.90	4.23	4.25	4.28	4.31	4.34	4.36	4.39	4.42	4.45	4.47
2.00	4.50	4.53	4.56	4.58	4.61	4.64	4.67	4.69	4.72	4.75
2.10	4.78	4.80	4.83	4.86	4.89	4.91	4.94	4.97	5.00	5.02
2.20	5.05	5.08	5.11	5.13	5.16	5.19	5.22	5.24	5.27	5.30
2.30	5.33	5.35	5.38	5.41	5.44	5.46	5.49	5.52	5.55	5.57
2.40	5.60	5.63	5.66	5.68	5.71	5.74	5.77	5.79	5.82	5.85
2.50	5.88	5.90	5.93	5.96	5.99	6.01	6.04	6.07	6.10	6.12
2.60	6.15	6.18	6.21	6.23	6.26	6.29	6.32	6.34	6.37	6.40
2.70	6.43	6.45	6.48	6.51	6.54	6.56	6.59	6.62	6.65	6.67
2.80	6.70	6.73	6.76	6.78	6.81	6.84	6.87	6.89	6.92	6.95
$2.90	6.98	7.00	7.03	7.06	7.09	7.11	7.14	7.17	7.20	7.22

100 SHARES - $3.00 to $2.99

PRICE	EVEN	5¢	10¢	1/8	15¢	20¢	1/4	30¢	35¢	3/8	40¢	45¢
$3.00	7.25	7.39	7.53	.....	7.66	7.80	7.94	8.08	8.21	.....	8.35	8.49
3.50	8.63	8.76	8.90	.....	9.04	9.18	9.31	9.45	9.59	.....	9.73	9.86
4.00	10.00	10.14	10.28	.....	10.41	10.55	10.69	10.83	10.96	.....	11.10	11.24
4.50	11.38	11.51	11.65	.....	11.79	11.93	12.06	12.20	12.34	.....	12.48	12.61
5.00	12.75	12.89	13.03	13.09	13.16	13.30	13.44	13.58	13.71	13.78	13.85	13.99
5.50	14.13	14.26	14.40	14.47	14.54	14.68	14.81	14.95	15.09	15.16	15.23	15.36
6.00	15.50	15.64	15.78	15.84	15.91	16.05	16.19	16.33	16.46	16.53	16.60	16.74
6.50	16.88	17.01	17.15	17.22	17.29	17.43	17.56	17.70	17.84	17.91	17.98	18.11
7.00	18.25	18.39	18.53	18.59	18.66	18.80	18.94	19.08	19.21	19.28	19.35	19.49
7.50	19.63	19.76	19.90	19.97	20.04	20.18	20.31	20.45	20.59	20.66	20.73	20.86
8.00	21.00	21.10	21.20	21.25	21.30	21.40	21.50	21.60	21.70	21.75	21.80	21.90
8.50	22.00	22.10	22.20	22.25	22.30	22.40	22.50	22.60	22.70	22.75	22.80	22.90
9.00	23.00	23.10	23.20	23.25	23.30	23.40	23.50	23.60	23.70	23.75	23.80	23.90
$9.50	24.00	24.10	24.20	24.25	24.30	24.40	24.50	24.60	24.70	24.75	24.80	24.90

MINIMUM COMMISSION

When the total amount involved in a transaction is less than $10.00, the charging of any commission shall be at the member's discretion.

When the total amount involved is $10.00 or more but less than $50.00, the commission shall be discretionary except that a minimum of $2.00 shall be charged.

When the total amount is $50.00 or more, the minimum commission shall be $5.00 except in connection with a transaction in rights or warrants when the minimum commission shall be $1.00

RIGHTS AND WARRANTS

The commission on rights and warrants shall be the same rate as applies to corresponding securities selling at equivalent prices. (See Special Commission Rates below.)

BONDS AND DEBENTURES

(a) Government of Canada bonds maturing in more than one year: ¼% of the selling price, with a minimum of 25 cents per $100.00 of the principal amount;

(b) All other bonds and debentures: ½% of the selling price, with a minimum of 50 cents per $100.00 of the principal amount;

(c) Government of Canada treasury bills and Government of Canada bonds maturing within one year: the charging of a commission shall be at the member's discretion.

If a transaction in bonds, debentures or treasury bills takes place off-the-floor (over-the-counter), it may be on a net basis, i.e. without commission.

SPECIAL COMMISSION RATES

On this Exchange the charging of any commission on purchase or sale of a stock which has been called for redemption shall be optional and the minimum price spread in such stocks shall be 5 cents if selling under $10.00, and 12½ cents if selling at $10.00 or over.

When rights or warrants are selling under one cent, the charging of a commission and the amount thereof, if any, shall be at the member's discretion.

ONTARIO STOCK TRANSFER TAX

On Shares of Stock Selling at:
Less than $1.00 ..... 1/10 of 1% of value of sale

	Per Share
$ 1.00 to $ 5.00	1/4 cent
5.01 to 25.00	1 cent
25.01 to 50.00	2 cents
50.01 to 75.00	3 cents
75.01 to 150.00	4 cents

Over $150.00 - 4 cents plus 1/10 of 1% of value over $150.00 per share.

On Rights and Warrants: No transfer tax is payable.
On Bonds and Debentures:
For each $100.00 par value or fraction ..... 3 cents

ALL TRANSACTIONS FOR VALUE THREE DAYS AFTER PURCHASE OR SALE.

Example 1

Jacques Riches sold 230 shares of Okanagan Helicopters at $5.75. Calculate the net proceeds in

(a) Alberta (b) Ontario.

Solution

Sale of 230 shares @ $5.75	$1,322.50	
Less commission $\frac{230}{100} \times 14.81$	34.06	
Net proceeds in Alberta (no transfer tax)		$1,288.44
Transfer tax in Ontario: 1¢ per share		2.30
Net proceeds in Ontario		$1,286.14

Example 2

Albert Heines of Guelph, Ontario bought 2150 shares of Blue Star at $0.06 and sold them two months later for $0.11. Find the net gain or loss.

Solution

Cost of 2150 shares @ $0.06		$129.00	
Add commission: $\frac{2150}{1000} \times 3.50$		7.53	
Total cost of purchase			$136.53
Sale of 2150 shares @ $0.11		$236.50	
Less commission:			
$5 × 2.15	10.75		
Transfer tax $\frac{1}{10}$ of 1% of 236.50	.24	10.99	
Net income from sales			225.51
Net gain			$ 88.98

EXERCISE 2·7

1. George Dakota bought 40 shares of Dominion Bridge at $22½. Find the total cost of the shares.

2. Find the commission that must be paid on each purchase:
   (a) 200 shares at $9.75 per share
   (b) 350 shares at $22.36 per share
   (c) 56 shares at $7⅜ per share

3. Find the transfer tax on each sale:
   (a) 1000 shares at $0.64 per share
   (b) 200 shares at $2.53 per share
   (c) 4000 shares at $4.95 per share

4. Investigate "over-the-counter" buying of stock. Write a brief report on this type of buying.

5. Mrs. D. J. Monds bought the following stock at the closing prices for December 24, 1971 (see Exercise 2·3, question 1). Calculate the total cost of the stocks.

A.G.F.M.	200 shares
Algo Cent.	100 shares
Atco Ind.	100 shares
Bay Mills	1000 shares

6. Mrs. I. R. Fast sold 300 Bank of Montreal shares at $\$18\frac{1}{2}$ and 100 Bank of Nova Scotia shares at $\$30\frac{1}{4}$. Find the net proceeds.

7. Mr. Wheatley sold and bought the following stocks on December 22, 1971 at the closing prices shown in Exercise 2·3, question 2. Calculate the amount in excess or the amount he had to put in.

Sold:

Parkland Bf	200 shares
Bashaw Led	1000 shares
Alta Copper	1000 shares

Purchased:

Arctic	1000 shares
Blue Glch	400 shares
Dogs Suds	200 shares
Cominco	100 shares

2·8 How Stock Prices Move

Some Things That Help You Predict the Market

The first thing to do is to study several daily newspapers or financial news. Important items on the financial page, in addition to prices, are the averages which serve as *barometers* of the business. Each average indicates the average movement of that exchange to which it refers. The Dow-Jones Index (one for Industrials, one for Rails, one for Utilities, and a composite one for all aspects of the market) is an American Index. In Canada each stock exchange produces its own index. The Toronto Stock Exchange, being one of the largest in Canada, is of most significance.

Economic future bright for B.C.

VANCOUVER ..... British Columbia businessmen paint a bright picture for the provincial economy in 1972 but are concerned over labor negotiations, a survey prepared by the provincial trade department showed Wednesday.

Executives of 250 B.C. business firms took part in the annual survey, which was released by Trade Minister Waldo Skillings.

His report said the forecast of growth "follows a year of very satisfactory performance after the relatively poor growth record of 1970". Preliminary statistics for 1971 indicate a significant recovery in the province's economy in the latter part of the year which is expected to carry on into 1972.

The executives also predicted easier access to capital and lower interest rates, which are expected to stimulate industrial expansion and reduce operating costs.

Employment rates are expected to rise but unemployment in the province is also forecast to continue at a high rate, which will mean the government's economic policies will continue to expansionary.

The executives said, however, that despite high unemployment, many companies are unable to fill labor requirements.

Labor costs also are expected to continue to rise, but the businessmen forecast a slight moderation in the average annual increase. The survey showed wage rates rose an average of 9.1 per cent in 1971.

A total of 234 firms reported they had spent $987 million on capital expenditures this year and expected to spend another $812 million in 1972.

Where some prices are headed now

Foreign cars ..... +5%*

North American cars ..... UP

Imported radios, TV sets and stereo components ..... +10% - 15%

Imported shoes .... +2% - 5%

Foreign motorcycles ..... +0% - 6%

Imported fabrics ..... UP

U.S. plastic resins .... Perhaps DOWN slightly

Domestic steel products ... Perhaps UP by the fall

Pharmaceuticals .... +2% - 3%

*Already increased 10% in the fall.

THE MARKET TREND

CALGARY -- The market remained down in late dealings today. Trading was light.

NEW YORK -- Stocks were lower in late afternoon activity today. Trading was active.

TORONTO -- Prices remained on the downside in late afternoon activity. Trading was moderate.

MONTREAL -- Banks and industrials led the market moderately lower today. Trading was light.

LONDON -- Stocks closed lower today for the first time this week. Trade was moderate.

VANCOUVER -- Trading was light during the early sessions today. Prices were well-mixed.

By studying the TSE index we notice that Industrial averages in late December, 1971 were up (183-184) compared to the previous year (173.01) and a little under the 1971 high (188.94).

DOW-JONES U.S. EXCHANGE INDEX

	30 Ind.	20 Rails	15 Util.	65 Stocks
Dec.27 ..	881.47	238.53	113.50	304.56
Dec.24 ..		HOLIDAY		
Dec.23 ..	881.17	238.88	112.38	304.17
Dec.22 ..	884.86	239.80	112.15	305.11
Dec.21 ..	888.32	241.13	112.41	306.38
Mo. Ago .	829.73	220.61	109.44	286.33
Yr. Ago .	830.91	164.75	120.27	267.89
1971 High	950.82	248.33	128.39	318.44
1971 Low	797.97	169.70	108.03	270.18

TORONTO STOCK EXCHANGE INDEX

	153 Ind.	12 Golds	29 B.M.	19 W.O.
Dec.27 ...		HOLIDAY		
Dec.24 ...	183.50	143.40	77.20	217.26
Dec.23 ...	183.06	143.46	77.10	217.80
Dec.22 ...	184.12	143.50	76.96	219.66
Dec.21 ...	183.76	143.07	76.51	220.25
Mo. Ago ..	165.58	152.50	70.23	194.76
Yr. Ago ..	173.01	161.42	89.32	197.54
1971 High.	188.94	202.66	103.01	242.60
1971 Low .	158.33	132.40	67.86	180.30

NEW YORK STOCK EXCHANGE INDEX

	1250 Comp.	1000 Ind.	76 Trans	136 Util.	75 Fin.
Dec.27 ...	55.71	59.93	48.90	38.22	72.77
Dec.24 ...		HOLIDAY			
Dec.23 ...	55.62	59.91	49.14	37.79	72.65
Dec.22 ...	55.85	60.27	49.34	37.56	72.81
Dec.21 ...	56.16	60.65	49.53	37.61	73.20
Mo. Ago ..	51.47	54.93	44.54	36.60	69.51
Yr. Ago ..	49.63	52.19	34.36	40.82	63.70
1971 High	57.76	61.95	49.59	43.56	75.00
1971 Low	49.60	52.31	35.39	36.09	63.88

MONTREAL STOCK EXCHANGE INDEX

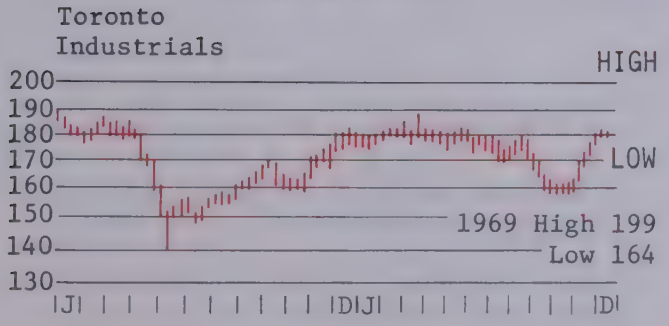
	65 Ind.	13 Util.	7 Banks	8 Papers	85 Comp.
Dec.27 ..		HOLIDAY			
Dec.24 ..	188.09	157.34	231.24	81.05	187.03
Dec.23 ..	188.25	156.98	231.57	81.07	187.10
Dec.22 ..	188.94	157.13	233.80	81.02	187.83
Dec.21 ..	188.86	156.97	236.04	80.99	187.96
Mo.Ago ..	169.00	145.07	211.31	68.49	169.15
Yr.Ago ..	175.85	151.86	176.21	91.30	171.93
1971 High	188.94	161.39	236.04	95.35	187.96
1971 Low	162.59	142.11	172.04	64.73	163.80

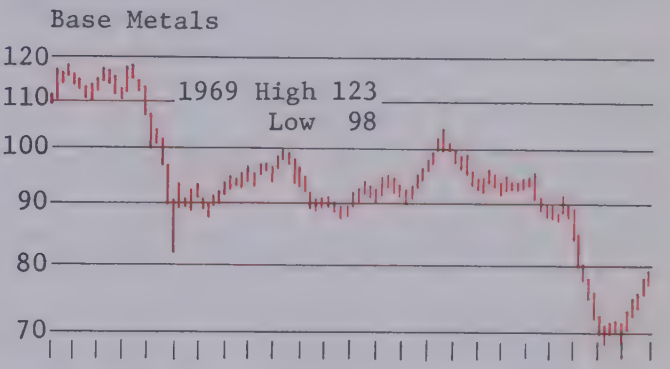
Graphs are often easier to interpret than columns of figures. Study these graphs.

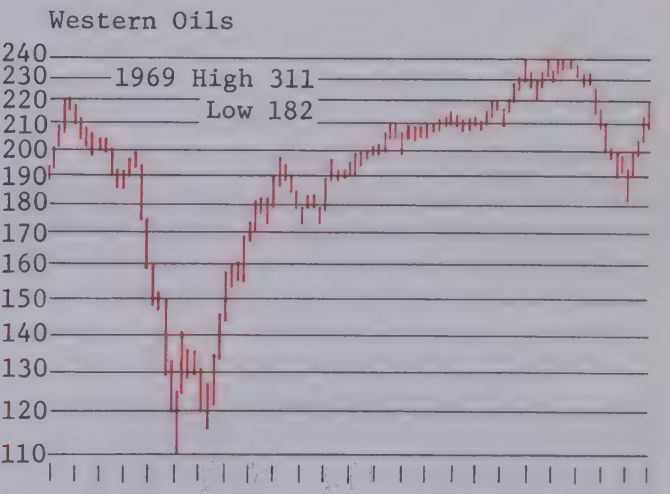
HOW STOCK PRICES MOVE

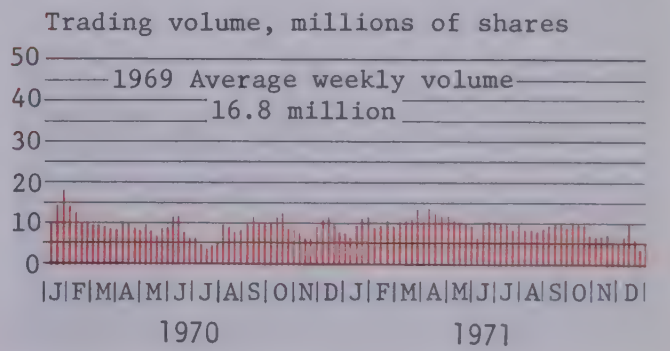
Weekly range

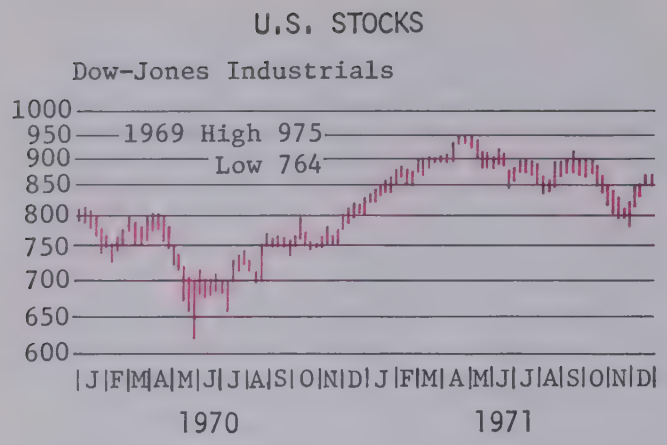
CANADIAN STOCKS











Discuss:

1. Do you agree these graphs are easier to interpret than the figures in the previous chart?
2. Do you think there is a time of the year that is best for selling on the TSE? for buying?
3. What would you think happened to stock prices in January 1972?
4. What happened in late December of 1971 with reference to Golds in the world monetary system?
5. In the fall of 1971 the Federal Government placed restrictions on oil exports to the U.S.A. Is there a reflection of this in the TSE averages? Explain.
6. In the fall of 1971 the United States Government imposed a surcharge (tax) on Canadian manufactured goods going into the United States. What affect did this have on the Industrial averages in Canada? In December of 1971 this surcharge was removed. Is there a reflection of this removal?

One important way of making money on the stock market is by buying stocks at one price and selling at a higher price (if a dividend is paid while one has the stock so much the better). In order to know which stocks might continue to go up you need to know the *trend* or have some information that might indicate a demand for the stocks and hence a price increase.

Often the newspapers carry items such as "NEW GAS WELL BLOWS IN FOR A B OILS" or "CANADA DOLLAR DROPS".

The novice should watch for news items and keep a table of values and a graph of the high and low prices of shares for his favourite stock.

Do questions 1-6 in Exercise 2·8.

In normal stock market transactions a person buys stock then sells it. Sometimes a person will sell stock (which he doesn't have but which he *borrows* from his broker) then he buys it back later. The broker uses the proceeds of the sale as collateral for the borrowed stock. This type of transaction is called *short selling*. The buyer is expecting the stock to decline in market value. He may sell at $5.00 a share and buy at $4.00 a share – making one dollar a share gross profit. Sometimes the person loses badly!

A buyer may feel that a particular stock will go up but he may not have enough funds to buy the stock. He can arrange with his broker to *buy on margin*. In effect, this means he borrows from the brokerage house a portion of the money required to buy the stock. He must pay interest on the money borrowed. Obviously, the buyer must have adequate capital in cash or collateral to cover the loan in case the shares drop instead of increasing.

EXERCISE 2·8

1. What do we mean when we say ". . . . serves as the barometer of the business"?
2. What can you say about the TSE average for Base Metals (BM) for December 21-24?

3. Do the TSE and MSE reflect the same trends in Industrials? Explain.

4. Do the Dow-Jones and TSE reflect the same trends for Industrials?

5. Discuss these questions with reference to the MSE averages. Assume the date of discussion is January, 1972.
   (a) If you had bought into paper stocks in 1970 what would you do now?
   (b) Do you think this would be a good time to buy into Bank stocks?
   (c) What do you think caused a drop in trading volumes of Industrials in the fall of 1971?
   (d) Can you think of anything that might cause a serious fall off of trading in paper in 1970 and 1971?
   (e) Generally, would it be a good time to buy into stocks in January 1972? Why?

6. Construct a table of 5 stocks you think would be good investments; construct a table and graph of 5 weekly high and low sales prices.

		HIGH	LOW	CLOSING
WEEK	STOCK	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
1				
2				
3				
4				
5				

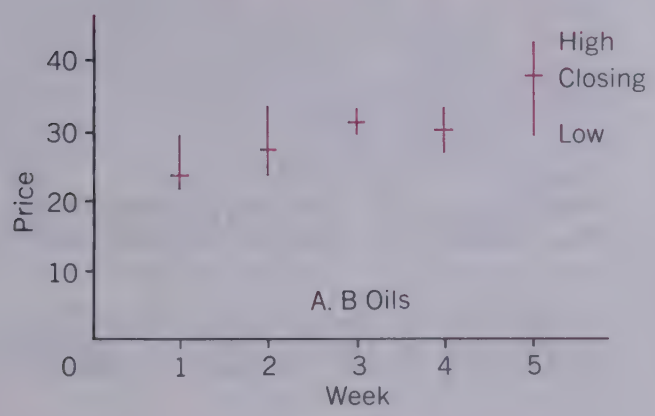
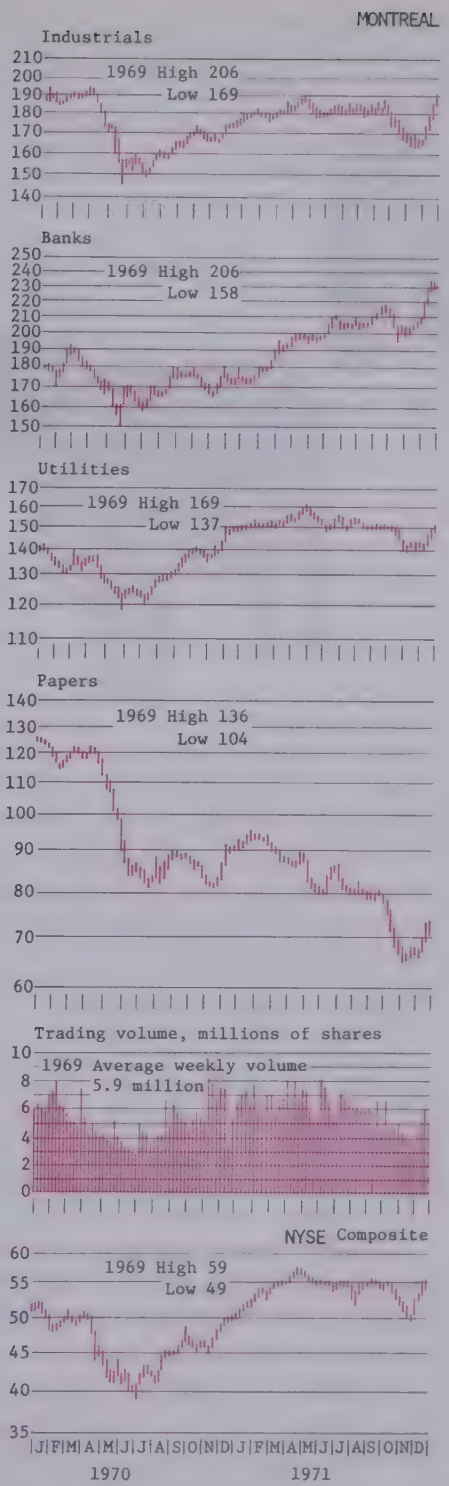


Figure 2-3



After studying the graphs decide which of the 5 stocks would likely be the best investment on a purchase and sell basis for a quick profit.

7. What is a Bear market?

8. What is a Bull market?

9. Keep a record of 5 successive stock market averages (one of Dow-Jones, TSE, MSE, NYSE, etc.). Show by means of a graph the trend in four of Industrials, Rails, Utilities, Golds, Base Metals, Western Oils, Papers. Show the volume trends for the five recordings. Describe the trends.

2 p.m. stock averages

(Canadian Dow-Jones)

NEW YORK STOCK EXCHANGE

Industrials	890.05	off	3.61
Rails	242.42	up	.95
Utilities	116.72	up	.19
Stocks	309.07	off	.19

Volume: 10,570,000

TORONTO STOCK EXCHANGE

Industrials	181.16	off	1.27
Golds	140.37	off	1.40
Base Metals	78.26	up	.14
Western Oils	214.56	off	.87

Volume: 1,441,000

MONTREAL STOCK EXCHANGE

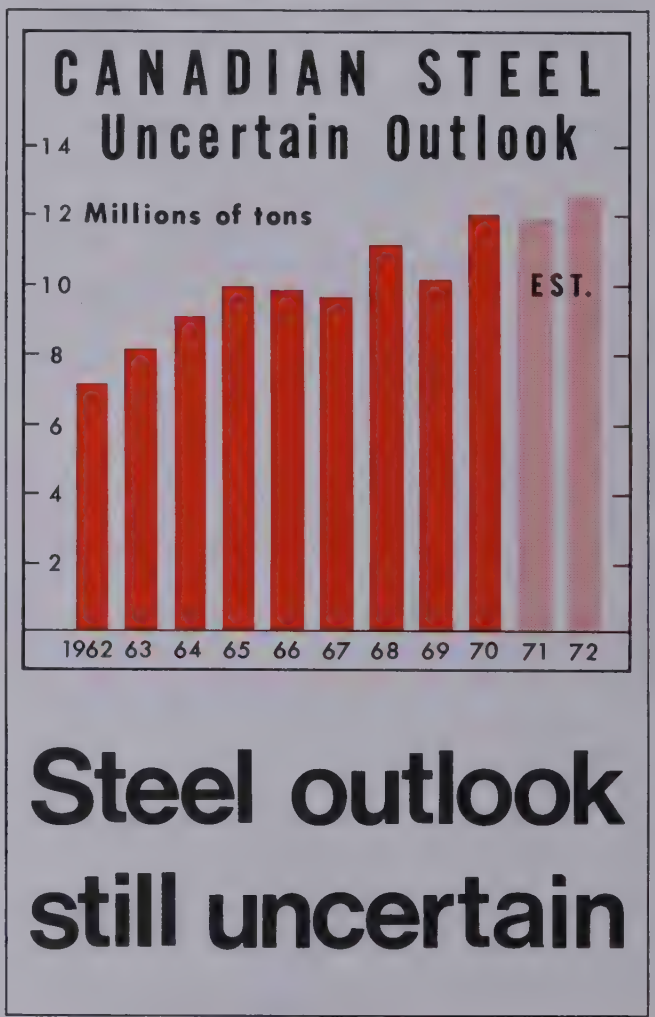
Industrials	186.61	off	1.17
Utilities	155.57	up	.12
Banks	227.33	off	1.75
Papers	81.62	off	.82
Composite	185.24	off	.97

Volume: 579,400

10. Choose 5 industrial stocks which you think are representative of the type. For ten days determine the average closing of the 5 stocks. Also obtain the TSE average for industrials for the ten days. On a large graph on the bulletin board show the two sets of averages. How do the averages of your stock compare to the averages of TSE Industrials? Do they both have the same trends?

11. What type of market trend would encourage a buyer to
    (a) sell short?
    (b) buy on margin?

12. From newspapers collect as many articles as you can in one week that would indicate trends in stocks. This is an example.



2·9 Bonds

A bond is a promissory note – an IOU. Let us consider the Inkless Pen Company again. It may decide it wants $10,000 for renovation and modernization of its plant. Instead of issuing more stock it decides to borrow the money – to obtain a loan. The company produces and sells 100 bonds each worth $100 (*face value or denomination*). Bonds usually carry a set rate of interest in excess of 3%.

Bondholders are not part owners of the company. They are only creditors of the company. If a company sells out, it must pay off the bondholders before the stockholders. Hence bonds are considered to be comparatively safe investments – not necessarily overly lucrative. They are for the conservative investor.

Once a bond is issued it may be sold and resold. The *price* a buyer is willing to pay for a bond may vary, but not as much as for stocks.

The date the loan is to be paid back is the *maturity date* of the bond.

Government bonds, federal, provincial and municipal, are the main type. Corporation bonds are sold by private companies.

Bonds are of two types: *bearer* and *registered*. A *bearer bond* is a certificate with coupons attached. The sum of all the coupons is the total interest payable on the bond. The coupons are dated and can be "clipped" on the dates indicated. A 5%, $100 bond would have $5.00 worth of coupons for each year. When a coupon is clipped, it can be taken to a bank and cashed for its face value.

The owner of a *registered bond* is recorded with the issuer and receives his interest payments as cheques as it comes due. The holder of a bearer bond is presumed to be the owner – no record of ownership is kept.

All bonds are quoted on the basis of $1000 par value. When a bond is quoted at 97 this means 97% of par value. Hence, 97 means 97% of $1000 or $970 for a bond of face value of $1000 or $485 for a bond of face value of $500.

Corporation bonds & debentures

(Supplied by the Investment Dealers' Association of Canada and several major investment dealers)
Approximate market prices at Dec. 24, 1971

Bonds	Int. Rate %	Due	Bid	Ask
Abitibi Pwr.& Paper ..	7¼	'87	89	91
Do.	9¾	'90	105	107
Acadia P.& P.	6¾	'84	82	85
Acklands	7½	'88	85	87
Alta. Gas Trunk	7½	'90	126½	128½
Do.	9¼	'90	106	108
Do.	9¾	'90	108	110
Do.	5¼	'81	85	87
Alta. Nat. Gas.	8½	'92	101	103
Algoma Cent. Rlwy.	8¾	'91	102	104
Algoma Steel	8¾	'91	101½	103½
Do.	5¼	'78	[illegible]	90
Allarco Dev.	7	'89	65	70
Aluminum Co.	4½	'73	98	99
Do.	9⅜	'91	107½	109½
Anglo-Cdn. P.& P.	6½	'86	84	86
Anglo-Cdn. Telephone	6½	'83	92	...
Argus Corp. Ltd.	5½	'75	92	93
Asbestos	9¾	'90	105	107
Auto Hardware	[illegible]	'89	70	74
Bank of B.C.	7½	'91	100½	101½
Bank of Nova Scotia	7	'87	93	95
Do.	6¾	'78	98	99
Do.	7	'91	100	100½
Bathurst Pap.	6	'84	71	76
Beaver Lumber	5¼	'89	95	...
Bell Tel. of Canada	5	'72	99	100
Do.	6	'81	90	92

Bonds	Int. Rate %	Due	Bid	Ask
Eaton Realty	6¼	'86	87	[illegible]
Eddy Match	6½	'86	86	88
Emco	7	'91	100	102
Extendicare	8½	'84	94	97
Falconbridge	7¾	'91	96	98
First Nat. Mtge.	7¼	'82	[illegible]	[illegible]
Do.	7	'80	90	92
Do.	6¾	'78	90	92
Do.	6½	'79	88	90
Four Seasons	8¼	'96	98	100
A. J. Freiman	6¼	'81	85	...
Fruehauf	5¼	'78	[illegible]	94
Gaz. Metro	5¾	'85	75	...
Do.	6	'88	78	...
Do.	7	'90	82	...
Gen. Motors Accept.	7	'86	90	92
Do.	8¼	'74	100	102
Do.	6½	'87	84	[illegible]
Gesco Distrib.	8	'84	82	[illegible]
Gt. Lakes Pap.	8	'89	[illegible]	97
Great Lakes Pwr.	4¼	'75	[illegible]	91
Do.	5¾	'77	89	91
Gt. Nor. Cap.	5	'79	82	87
Gt. Pac. Ind.	9	'85	[illegible]	...
Grouse Mtn.	7	'86	[illegible]	...
Gulf Oil	7⅛	'88	96	97
Do.	8½	'89	105	107
Do.	8½	'90	105½	107½
Halifax Dev.	8	'93	[illegible]	81
Household Fin.	8	'92	[illegible]	[illegible]
Hud. Bay Accept.	9¼	'90	104	106
Do.	8¾	'91	103	104
Hudson Bay M.& S.	9	'91	103½	105½
Husky Leasebacks	6¼	'85	85	...

Note on the quotations that Alta. Nat. Gas bonds have an interest rate of $8\frac{1}{2}$%, maturity date 1992. The *Bid price*, meaning the price you could sell it for, is 101% while the *Ask price*, the price you would have to pay to buy it, is 103%. The difference between the Bid and Ask price is usually the broker's commission. In the case cited, the buyer pays more than the par value. Hence it is said to be at a *premium*. If the Bid price were less than 100% the bond is said to be *discounted*

The accrued interest is the interest that is due since the last interest was paid.

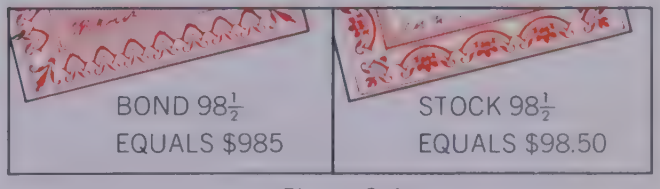


Figure 2-4

Example 1

A man, on March 31, 1972, bought a $1000, 5% Bell Telephone of Canada bearer bond at 99 due on December 31, 1972. Interest dates are July 1 and December 31. Find the cost of the bond.

Solution

He must pay 99% of par value of the bond plus accrued interest.

Purchase price $= .99 \times \$1000$ or $990.

Accrued interest: $I = prt$

January 31 $= 1000 \times .05 \times \frac{90}{365}$

February 28

March 31

90 days $= \$12.33$

The total cost is $990 + $12.33 = $1002.33

EXERCISE 2·9

1. Why do you think a bond would be at a
   (a) premium? (b) discount?

2. Calculate the price paid for the bonds listed if Ask price is paid. Use the current date on which you work the questions.

Government of Canada bonds

Dec. 24, 1971

	Bid	Ask	Yield
3 Canada ... Sep prep.	42.00	44.00	6.81
3¾ Canada ... Mar 98	61.00	63.00	6.79
6½ Canada ... Apr 72	100.60	100.70	3.95
7¼ Canada ... Apr 72	100.85	100.95	4.38
4¼ Canada ... Sep 72	100.15	100.20	3.94
5¼ Canada ... Dec 72	101.15	101.25	3.91
8 Canada ... Feb 73	103.80	103.90	4.40
7 Canada ... Feb 73	102.65	102.75	4.45
7 Canada ... Apr 73	103.05	103.15	4.45
6½ Canada ... Jul 73	102.90	103.00	4.46
6¾ Canada ... Jul 73	103.30	103.40	4.44
5 Canada ... Oct 73	100.80	100.90	4.47
5¼ Canada ... Dec 73	101.40	101.50	4.43
6¼ Canada ... Dec 73	103.20	103.20	4.47
5 Canada ... Apr 74	101.20	101.30	4.39
7¼ Canada ... Apr 74	105.80	105.90	4.50
[illegible] Canada ... Jun 74	103.40	103.50	4.50
[illegible] Canada ... Oct 74	108.75	109.00	4.49
4¼ Canada ... Dec 74	99.75	99.85	4.30
5½ Canada ... Dec 74	102.55	102.65	4.53
7¼ Canada ... Apr 75	106.25	106.75	5.00
5¾ Canada ... Dec 79	100⅜	100⅝	5.65
6¼ Canada ... Apr 80	102.50	103.00	5.79
5½ Canada ... Aug 80	98.50	99.00	5.65
4½ Canada ... Sep 83	87.25	87.50	6.00
5 Canada ... Jun 88	84.00	85.00	6.49
6¾ Canada ... Feb [illegible]	101.00	101.50	6.00
5¼ Canada ... May 90	86.50	87.00	6.47
5¾ Canada ... Sep 92	91.50	92.50	6.41
6½ Canada ... Oct 95	98.50	99.00	6.58

CDN. NATIONAL RLWY. --CANADA GUARANTEED

	Bid	Ask	Yield
5½ C.N.R. ... Dec 71	100.05	....	...
3¾ C.N.R. ... Feb 74	98.90	99.00	4.24
5 C.N.R. ... May 77	96.75	97.50	5.53
4 C.N.R. ... Feb 81	84.75	85.50	6.08
5¾ C.N.R. ... Jan [illegible]	92.50	93.50	6.49
5 C.N.R. ... Oct 87	84.50	85.50	6.47

PROVINCIAL AND PROVINCIAL GUARANTEED

	Bid	Ask	Yield
8⅞ Pr. Alta.[9] .. Apr 88/90	108	...	....
7⅞ Pr. Alta. ... Oct 89/91	101¼	101¾	7.45
7½ Alta.G.Tel. . Dec 87/91	99⅜	99⅞	7.51
5¾ B.C. Elec. .. Apr 77	94	96	6.62
5 B.C. Elec. .. Feb 82	83	85	7.10
4¾ B.C. Elec. .. Aug 86	77	79	7.08
5 B.C. Elec. .. Aug 86	79	81	7.10
5½ B.C. Elec. .. Aug 86	83	85	7.15
8½ Man. Hyd. ... Sep 89/91	106½	107½	7.72
8 Man. Hyd. ... Apr 89/91	102	103	7.68
7⅞ Man. Tel. ... Nov 89/91	101	102	7.67
8 Man. Tel.[1] .. Sep 74	104[1]	...	....
9½ Nfld.[2]..... Mar 90	109	...	....
7½ Nfld. ..... Feb 88	91	...	....
6¾ N.B. ..... Aug 90/92	89¼	...	....
7¾ N.B. ..... Mar 94/96	99¾	100¼	7.72
5½ N.S. ..... Mar 84	83	...	....
6¼ N.S. ..... Nov 88/91	85	...	....
7⅝ N.S. ..... Dec 89/91	98½	99	7.72
7¼ N.S. ..... Mar 89/91	95	97	7.55
9¼ Pr. Ed. Is. . Nov 90	108	...	....
7½ Quebec[3] ..... Dec 72	101	...	....
7¾ Quebec[4] ..... Dec 74	101½	...	....
9¼ Quebec[5] ..... Jun 77	107	...	....
9¼ Quebec[10] .... Oct 78	107	...	....
7½ Quebec ..... Apr 81	99	...	....
5¼ Quebec ..... Jul 81/83	81	...	....
6 Quebec[6] ..... Dec 79	90	...	....
5½ Quebec ..... Jun 84/86	80	...	....
5½ Quebec ..... Jan 85/89	75½	...	....
8 Quebec ..... Feb 90/94	100½	101½	7.85
9½ Quebec ..... Oct 92/95	108	...	....
7½ Quebec ..... Sep 90/93	94	...	....
7¾ Quebec ..... Dec 93/96	97⅜	98⅜	7.91
7¾ Que. Hyd.[7] .. Mar 74	101	...	....
9½ Que. Hyd.[8] .. Feb 75	101½	...	....
6½ Que. Hyd. ... Feb 87/90	84½	...	....
6 Que. Hyd. ... Oct 90/94	77½	...	....
7 Que. Hyd. ... Oct 90/94	88½	...	....
8¾ Que. Hyd. ... Jul 93/96	107	...	....

(a) Government of Canada Bonds

Maturity dates:		
	April 73	$1000
	April 74	$5000
	August 80	$2000

(b) C.N.R.

Maturity dates:		
	Feb. 74	$500
	Feb. 81	$300

(c) Alta. G. Tel. $200

(d) Man. Tel. 6 at $500 each

(e) Pr. Ed. Is. $2500

3. Calculate the price paid for a $2000 Government of Canada bond maturing June 1, 1974, purchased April 30, 1972, if dates of the 7 coupons are July 1 and January 1.

4. Calculate the price paid for each of the following if coupon dates are July 1 and January 1.

Purchase Date	*Face Value*	*Bond*	*Maturity Date*
Oct. 1, 1974	$3500	5% Can.	June 1, 1988
June 1, 1973	$2100	5½% Can.	Dec. 15, 1974
Nov. 1, 1975	$4300	5¾% C.N.R.	Jan. 15, 1985

5. For how many days will there be accrued interest on each:

	Date of Last Interest Payment	*Date of Purchase*
(a)	May 15, 1973	July 4, 1973
(b)	Feb. 10, 1974	June 15, 1974
(c)	Sept. 22, 1974	Dec. 30, 1974
(d)	Dec. 6, 1974	Feb. 15, 1975

2·10 Approximate Yield on a Bond

We define yield on a bond in a manner similar to that of stocks:

$$\text{Yield} = \frac{\text{average yearly income}}{\text{average price}}$$

This gives an approximate yield usually accurate to one decimal place.

Average income and average price are calculated as shown in the next example.

Example 1

A man purchases a $1,000 bond at 98 due in 10 years. It carries $7\frac{1}{2}\%$ interest coupons.

Solution

The purchase price (.98 × $100)	$ 980
The redemption price	$1,000
The gain	$ 20

The average gain over 10 years is $\frac{20}{10}$ = $2.00 per year.

The average yearly income from the bond is 75.00 + 2.00 = $77.00 (interest + gain).

The average price of the bond purchased at $980 and redeemed at $1000 is

$$\frac{980 + 1000}{2} = \$990.$$

$$\text{Yield in \%} = \frac{\text{average yearly income}}{\text{average price}} \times 100\%$$

$$= \frac{77}{990} \times 100\%$$

$$= 7.8\%$$

EXERCISE 2·10

1. If the following bonds are bought in 1973 at Ask price, find the approximate yield for each:

Canada	5	1988	Bid 84	Ask 85
CNR	$5\frac{3}{4}$	1985	Bid 92.50	Ask 93.50

2. Calculate the approximate yield on each of the stocks shown below if the bonds are purchased in 1971, if the Bid price is paid. Check your answers with the stated yield. (Some answers may vary more than one decimal.)

KEY BOND ISSUES

		December 10,	1971 Bid	Change on week	% Yield
Canadas					
	$4\frac{1}{4}\%$	Sept./72	100.13	+0.01	4.05
	$5\frac{1}{4}\%$	Dec./73	101.40	-0.05	4.50
	$4\frac{1}{2}\%$	Sept./83	87.50	-0.13	6.00
Provincials					
Ontario	$7\frac{1}{4}\%$	Dec./96	99.63	+0.13	7.27
Quebec	$7\frac{3}{4}\%$	Dec./96	97.75	-0.25	7.96
Corporates					
Distillers Corp.	$7\frac{5}{8}\%$	Dec./91	99.25	-	7.70
Alberta Gas Trunk	8%	Dec./91	100.00	-	8.00

3. Compare the yield with the interest rate on all bonds in question 2 when
   (a) selling price is at a premium
   (b) selling price is discounted.
   What relation is indicated?

4. Examine a daily paper and select one investment opportunity that you would rate as good. Explain your choice.

5. Find out if there are any bonds outstanding on any schools in your community. What rate of interest is being paid? On what date will they mature? What provision has your school board made to pay them off?

Using the Library

A delightful book of short stories is *Fantasia Mathematica* by Clifton Fadiman (Simon and Schuster, New York, 1958).

Obtain this book and read the selections that appeal to you.

There are several selections on topology. You may wish to find out what a Moebius Strip is (if you don't know) before you read *A Botfly and the Moebius Strip*, *A Subway Named Moebius* and *The Unfortunate Topologist*.

Science Reading And He Built A Crooked House research to find a simple explanation of the fourth dimension (sometimes a cubarium illustrates the fourth dimension).

An extraordinary science fiction (including a description of the school of the future) is titled *John Jones's Dollar*.

Read on and enjoy this delightful book.

2. Prepare a set of illustrations with captions or a written illustrated report on the following. It is suggested the class be divided into groups so that most topics are covered.
   (a) The history of time and the way man has kept track of it, i.e. the year, the calendar, units, sundials, primitive, etc.
   (b) The history of flight from early attempts of flying like birds to the modern day flights to other planets.
   (c) The history of mathematics in medicine, psychology, sociology, etc.
   (d) The history of mathematics in games, i.e. from old ones like NIM to modern day games. You may wish to start by analysing games on the market today. Which ones are based on mathematical concepts or principles? How many are really ancient games with a modern setting? Don't forget the game of billiards. You may even wish to build an elliptical billiard table.
   (e) Mathematics in nature, i.e. Fibonacci series, Golden Ratio, in genetics, theory of heredity, etc.
   (f) Mathematics in architecture dating from historical times to modern buildings.
   (g) Mathematics in art (in addition to that studied with respect to conic and projective geometry).
   (h) Mathematics and music, mathematics and literature. (See Volume 4 of The World of Mathematics for a start.)

REVIEW EXERCISE 2

1. J. March bought $1000 worth of shares for $850. The dividend in one year was $350. What is the percent return on this investment?
2. What are the advantages of buying
   (a) preferred stocks?
   (b) common stocks?
3. M. M. Jay bought $2500 worth of stocks. The dividend was 6%. He paid $2800 for the stock. What was the percent return on the investment?
4. What is the difference between par and no-par-stocks?
5. Calculate the dividend yield and P/E Ratio for Loblaw Companies Limited (A and B) and Industrial Minerals. Use closing prices.

Stock	Sales	High	Low	Close	Change	High	Low
Costain R	10290	$7 1/2	7 1/4	7 1/4		7 7/8	465
Craig Bit	100	475	475	475	-25	5 1/4	350
Crain R L	4655	$12	11 1/8	11 1/8	+ 1/8	16 1/2	9 3/4
Imp Life	12	$137	137	137	+1 1/2	162	130
Imp Oil	172277	$32	31 1/8	31 1/4	- 5/8	32 3/8	18 5/8
Income Di	z100	$5 3/4	5 3/4	5 3/4		9	425
Inc Dis w	670	60	58	58	-1	110	47
Indal Can	200	$8 3/8	8 3/8	8 3/8	+ 3/8	10 1/4	7 1/4
Industrial Minerals	2318	$9 3/4	9 5/8	9 5/8	+ 1/2	14	8 1/4
Ind Adhes	2900	$13 7/8	12 1/8	13 3/4	+81	15 1/4	11 1/2
Inglis	950	$10	9 1/2	10	+ 1/8	11 1/8	8 1/4
Inland Gas	8764	$13 1/8	12 1/2	13	+ 1/4	13 7/8	11
Inland G p	1388	$14 3/4	14 1/4	14 1/2	+ 1/2	16	13 1/2
Inter-City	13760	6 7/8	6 5/8	6 3/4	+ 1/8	7w6	5
Lincoln Tr	950	$12 1/4	11 3/4	12 1/8	+ 1/8	13 1/2	10
Livingsto	1520	$9 1/2	9	9	- 3/4	10 1/2	6 3/4
Living w	1872	535	490	520	+10	650	295
Lob Co A	4992	$6	5 1/2	5 3/4	+ 1/4	6	460
Lob Co B	14610	$6	5 1/2	5 3/4	+ 1/4	6	460
Lob Co pr	460	$30 1/2	29 3/4	30 1/2	+1	32 3/4	28 1/2
Lob G A pr	838	$19 3/8	18 1/2	19 3/8	+ 1/8	21 1/2	17 1/2
Lob G B pr	1685	$20	19 3/4	20	+ 1/4	21 3/4	18 3/4
Lob Inc	z25	$6 1/2	6 1/2	6 1/2		8 1/4	6 3/8
Loeb M	3905	360	340	350	+20	470	310

LOBLAW COMPANIES LIMITED

NOTICE is hereby given that a dividend has been declared on the capital stock of the company as follows:

Class "A" Shares ..... 9 1/2 cents per share

Class "B" Shares ..... 9 1/2 cents per share

The dividend will be payable December 31, 1971 to shareholders of record at the close of business on the 10th day of December 1971. The transfer books will not be closed. Payment will be made in Canadian funds.

By Order of the Board.

E. L. JONES, Secretary.

Toronto, November 30, 1971

industrial minerals limited

NOTICE OF DIVIDEND

The Directors of Industrial Minerals Limited have declared a dividend of twenty-five cents (25¢) per share, payable in Canadian funds on December 20, 1971, to shareholders of record as at the close of business December 7, 1971.

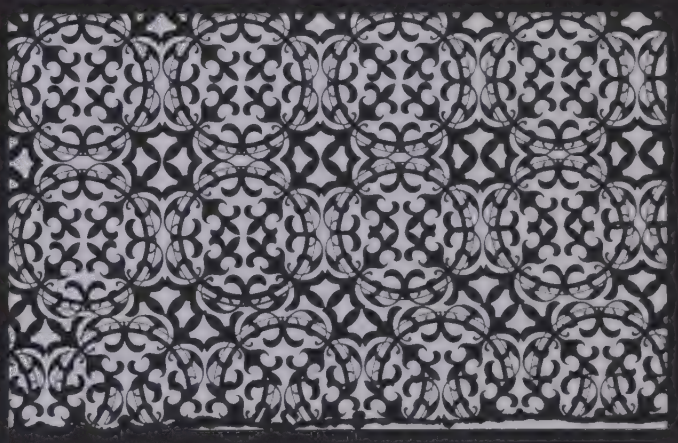
By Order of the Board

D. D. ANDERSON, Secretary

Toronto, Ontario
November 24, 1971.

Chapter Three

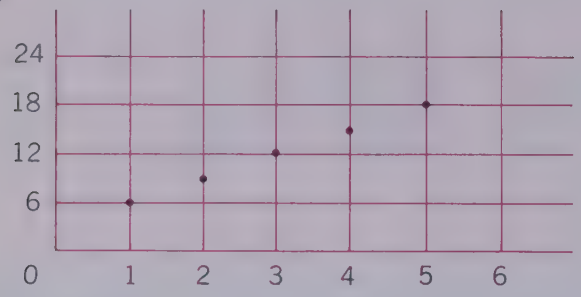
SEQUENCES AND SERIES



3·1 Patterns

Observe the following patterns.

(a)



(b) $+1, -1, +1, -1, +1, \ldots$

(c) $2, 5, 8, 11, 14, \ldots$

(d) ●■, ●●■, ●●●■, ...

(e) $2, 6, 18, 54, \ldots$

(f) $64, 32, 16, 8, \ldots$

(g) $-4, -12, -36, -108, \ldots$

(h) $17, 13, 9, 5, \ldots$

(i) $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$

(j) $\frac{15}{16}, \frac{3}{4}, \frac{9}{16}, \frac{3}{8}, \ldots$

(k) $\sqrt{12}, \sqrt{18}, \sqrt{24}, \sqrt{30}, \ldots$

(l) $1, 0.1, 0.01, 0.001, \ldots$

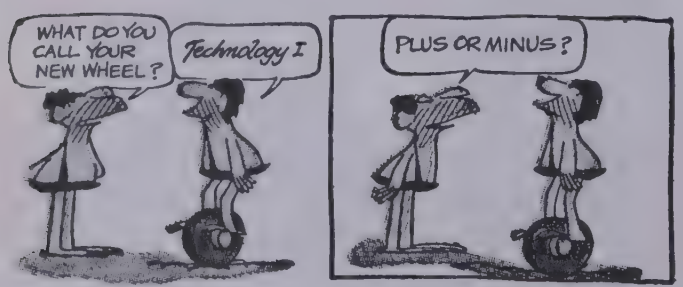
(m) $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots$

(n) O, T, T, F, F, ...*

**Hint!* Consider the natural numbers.

EXERCISE 3·1

1. For each pattern above try to determine what the next three terms might be.



3·2 Arithmetic Sequences

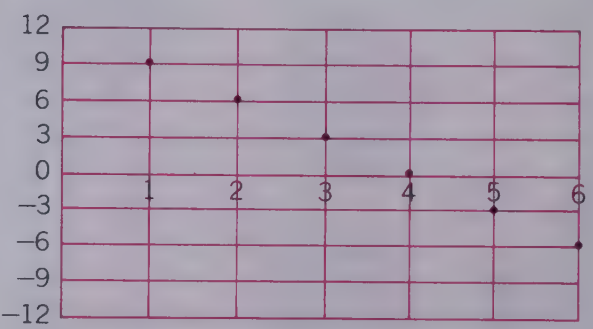
A pattern of numbers or other things associated with numbers (lengths, volumes, sizes, etc.) is called a sequence.

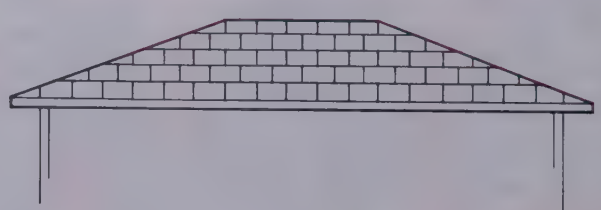
In each pattern in Section 3·1 a specific relation or rule can be observed. It is then possible to extend each pattern (sequence) indefinitely or until it terminates.

> A sequence in which each consecutive number in the pattern is derived by *adding* some specific *constant* to the previous number is called an *arithmetic sequence*.

Some books refer to an arithmetic sequence as an arithmetic progression (A.P.)

Thus, the following are examples of arithmetic sequences.

1. $7, 9, 11, 13, 15, \ldots$
2. $23, 15, 7, -1, -9, \ldots$
3. 

4. 

The constant added to any term of the sequence

$$7, 9, 11, 13, 15, \ldots$$

to arrive at the succeeding term is 2. This constant is called the *common difference*

EXERCISE 3·2

1. Of the patterns in Section 3·1, which are arithmetic sequences?
2. For each arithmetic sequence in question 1 what is the common difference?
3. Give three examples in the physical world that illustrate arithmetic sequences.

3·3 Any Term of an Arithmetic Sequence

For the arithmetic sequence:

$$-12, -5, 2, 9, 16, \ldots$$

the first term (t_1) is -12,
the second term (t_2) is -5,
the third term (t_3) is 2, and so on.

The common difference (d) is 7.

$$t_3 - t_2 = +7 \qquad t_2 - t_1 = +7$$

Finding the eighth term (t_8) of this sequence would not be difficult. Extending the sequence by adding the following terms:

$$23, 30, 37, \ldots$$

leads us to deduce that $t_8 = 37$.

However, to find the 42nd term (t_{42}) by the same method would prove time consuming. Let us study the sequence in this way.

$t_1 = -12$
$t_2 = -5$
$\quad = -12 + 7$ i.e. $t_2 = t_1 + d$

(3) $t_3 = 2$ i.e. $t_3 = t_2 + d$
$\quad = -5 + 7$ $\quad = (t_1 + d) + d$
$\quad = (-12 + 7) + 7$ $\quad = t_1 + 2d$
$\quad = -12 + 2(7)$

(4) $t_4 = 9$ i.e. $t_4 = t_3 + d$
$\quad = 2 + 7$ $\quad = (t_1 + 2d) + d$
$\quad = [-12 + 2(7)] + 7$ $\quad = t_1 + 3d$
$\quad = -12 + 3(7)$

$t_5 = t_4 + d$
$\quad = (t_1 + 3d) + d$
$\quad = t_1 + 4d$

Thus we reason that

$$t_{42} = t_1 + 41d.$$

By Substitution

$t_{42} = -12 + 41(7)$
$\quad = -12 + 287$
$\quad = 275.$

Using this method, find the 15th term (t_{15}) and the 101st term (t_{101}) of the same sequence.

Did your answers come out to

$$t_{15} = 86 \quad \text{and} \quad t_{101} = 688?$$

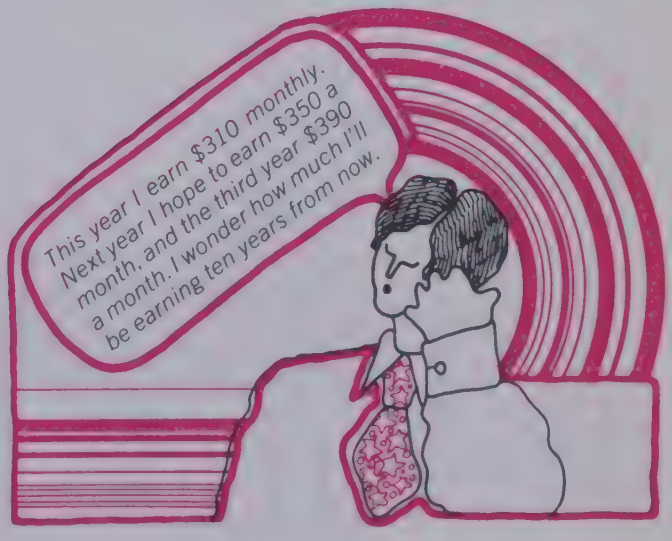
EXERCISE 3·3

1. For the arithmetic sequence

   $$2, 5, 8, 11, \ldots$$

   (a) find the common difference
   (b) find the twelfth (t_{12}) and seventeenth (t_{17}) terms using the method above. Then check your answers by extending the sequence.

2. If the pattern this young man has in his mind were extended for the 10 years, what would his monthly earnings be 10 years from now? (Check: Are there 10 or 11 terms in this sequence?)


This year I earn $310 monthly. Next year I hope to earn $350 a month, and the third year $390 a month. I wonder how much I'll be earning ten years from now.

3. Write the seven terms of each arithmetic sequence, following the given term:
   (a) $t_1 = 3$ and $d = 2$
   (b) $t_1 = 17$ and $d = -3$
   (c) $t_4 = 15$ and $d = -1$
   (d) $t_{40} = 12$ and $d = -4$
   (e) $t_3 = -64$ and $d = 2$
   (f) $t_1 = -100$ and $d = 5$
   (g) $t_{44} = -24$ and $d = 8$
   (h) $t_{10} = 100$ and $d = -5$
   (i) $t_2 = 400$ and $d = -8$
   (j) $t_{54} = 60$ and $d = 10$

   For each sequence find t_{52}.

4. For each of the following arithmetic sequences, find the eleventh and twenty-fifth terms.
   (a) 45, 53, 61, 69, ...
   (b) $1\frac{1}{2}, 2\frac{3}{4}, 4, 5\frac{1}{4}, \ldots$
   (c) 2.1, 2.3, 2.5, 2.7, ...
   (d) 80, 75, 70, 65, ...
   (e) 11.3, 8.7, 6.1, 3.5, ...

5. After a purchase in a store your budget account has a balance of \$126 on September 10. You are making payments toward it of \$15 on the first of each month. What will the balance be in your account on March 10 of the following year? What will be the balance of this same account after your fifth payment? (No interest calculated.)

6. In an arithmetic sequence
   (a) if $t_1 = 83$ and $t_3 = 71$, find d and t_2
   (b) if $t_2 = 99$ and $d = -12$, find t_1 and t_{12}
   (c) if $t_8 = 17$ and $d = 4$, find the tenth term
   (d) if $t_1 = 5.27$ and $d = -1.35$, find t_{17}
   (e) if $t_1 = 65$ and $t_3 = 72$, find d and t_2
   (f) if $t_3 = 8$ and $t_5 = 15$, find d and t_4
   (g) if $t_7 = 64$ and $t_9 = 68$, find d and t_1
   (h) if $t_{40} = 512$ and $t_{43} = 524$, find d and t_{41}, t_{42}.

3·4 The General Term of Arithmetic Sequences

Let us deal with arithmetic sequences in the general form. Take the sequence

$$t_1, t_2, t_3, t_4, t_5, \ldots$$

with the common difference d. Find the nth term (t_n) of this sequence.

$$\begin{aligned} t_1 &= t_1 \\ t_2 &= t_1 + d \\ t_3 &= t_1 + 2d \\ t_4 &= t_1 + 3d \\ &\vdots \quad\quad \vdots \\ t_{17} &= t_1 + 16d \\ &\vdots \quad\quad \vdots \\ t_n &= t_1 + (n-1)\,d \end{aligned}$$

The nth term,

$$\boxed{t_n = t_1 + (n-1)\,d}$$

is known as the *general term* of an arithmetic sequence.

Note! Because the sequence is

$$t_1, t_2, t_3, t_4, \ldots, t_n$$

then n is always a natural number, i.e., $n \in N$.

Example 1

Find the sixteenth term of an arithmetic sequence where the first term is -87 and the common difference is 16.

Solution

$$\begin{aligned} t_1 &= -87,\ d = 16 \\ t_n &= t_1 + (n-1)\,d \\ t_{16} &= -87 + (16-1)\,16 \\ &= -87 + 15\,(16) \\ &= -87 + 240 \\ &= 153 \end{aligned}$$

Example 2

In an arithmetic sequence, if $t_{31} = 181$ and $d = 6$, find t_2.

Solution

$$\begin{aligned} n &= 31 \\ t_n &= t_1 + (n-1)\,d \\ 181 &= t_1 + (31-1)\,6 \\ 181 &= t_1 + (30)\,6 \\ 181 &= t_1 + 180 \\ t_1 &= 1 \\ \therefore\ t_2 &= 1 + 6 = 7 \end{aligned}$$

Before proceeding do questions 1 through 6 of Exercise 3·4.

Example 3

In an arithmetic sequence $t_7 = 12$ and $t_{10} = 24$. Find the first three terms of the sequence.

Solution

$$\begin{aligned} t_{10} - t_7 &= (t_1 + 9d) - (t_1 + 6d) \\ 24 - 12 &= 3d \\ 4 &= d \\ t_n &= t_1 + (n-1)\,d \\ 24 &= t_1 + (10-1)\,4 \qquad (t_{10} = 24) \\ 24 &= t_1 + 36 \\ t_1 &= -12 \end{aligned}$$

Hence $t_1 = -12$, $t_2 = -8$, $t_3 = -4$.

Example 4

Which term of the sequence

$$5, 8, 11, 14, \ldots$$

is 80?

Solution

$t_1 = 5$, $d = 3$, $t_n = 80$, find n.

$$\begin{aligned} t_n &= t_1 + (n-1)\,d \\ 80 &= 5 + (n-1)\,3 \\ 80 &= 5 + 3n - 3 \\ 80 &= 2 + 3n \\ 3n &= 78 \rightarrow n = 26 \end{aligned}$$

80 is the 26th term of the sequence.

EXERCISE 3·4

1. Determine which of the following are arithmetic sequences. Find the common difference for each.
   (a) $0.78, 0.89, 1.00, 1.11, \ldots$
   (b) $a + r, 2a + r, 4a + r, 8a + r, \ldots$
   (c) $5\sqrt{3}, 5\sqrt{6}, 10\sqrt{3}, 10\sqrt{6}, \ldots$
   (d) $4\frac{1}{2}, 3\frac{1}{4}, 2, \frac{3}{4}, \ldots$
   (e) $2x - 3, 5x - 2, 8x - 1, 11x, \ldots$

2. Find the 41st term of
   $$3, 7, 11, 15, \ldots$$

3. If $t_{47} = -173$ and $d = -4$, find t_1, t_{15}.

4. The twelfth term of an arithmetic sequence is 82 and the common difference is 4. Find the third term.

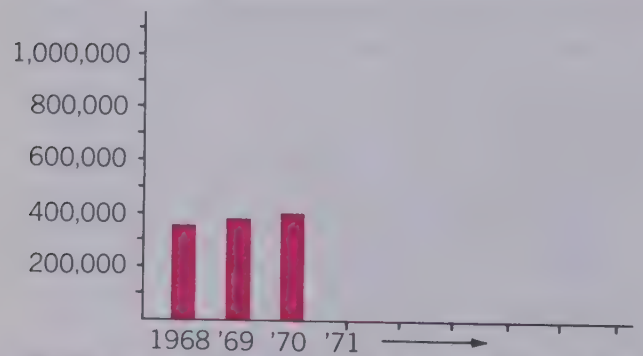
5. The ninth term of an arithmetic sequence is 56 and the tenth term is 59. Calculate the first term and the twentieth term.

6. If the fifth term of a sequence is 13 and the seventh term is 19, find the common difference and the first term.

7. For each of the following sequences, find the first three terms:
   (a) $t_{12} = 50,\ t_{15} = 65$
   (b) $t_{27} = -12,\ t_{31} = -18$
   (c) $t_9 = 19\sqrt{2},\ t_{14} = 29\sqrt{2}$
   (d) $t_{21} = 9.48,\ t_{26} = 10.58$

8. Find the 41st term of:
   (a) $\frac{1}{3}, \frac{11}{24}, \frac{7}{12}, \frac{17}{24}, \ldots$
   (b) $\frac{1}{5}, \frac{7}{10}, \frac{6}{5}, \frac{17}{10}, \ldots$
   (c) $\frac{1}{3}, \frac{7}{12}, \frac{5}{6}, \frac{13}{12}, \ldots$
   (d) $\frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \ldots$

9. In the recent past the city of Calgary, Alberta was one of the fastest growing cities in Canada on a percentage basis. At the end of 1970 its population was 398,034. For five years previous to 1970 its yearly average population increase was 15,000 people (approximately). If this trend of increase were to continue, in what year would the population of Calgary be one million?



10. Your class offered to raise $310 to buy microscope kits for a school in Nigeria. A canvass of the local businesses raised exactly $100. The 28 students of your class agreed to donate 5 cents each per school day until the total amount was raised. How many weeks will your class have to contribute to raise the required amount? (Use the general term to solve this problem.)

11. Which term of the sequence
   (a) 3, 9, 15, 21, ... is 117?
   (b) 4, −1, −6, ... is −141?
   (c) $\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$ is −2?

12. A sandbag ballast is dropped overboard from an ascending balloon. In the first second the sandbag falls 16 feet, in the next second it falls 48 feet, and in the third 80 feet. If this pattern of drop continues
   (a) in which second after release will it drop 336 feet?
   (b) how many feet will it drop in the 30th second?

3·5 Sum of an Arithmetic Series

This set of numbers

$$3, 5, 7, 9, \ldots, 21$$

is an *arithmetic sequence* with $t_1 = 3$, $d = 2$, $t_{10} = 21$.

This sum

$$3 + 5 + 7 + 9 + \ldots + 21$$

is an *arithmetic series* with $t_1 = 3$, $d = 2$, $t_{10} = 21$.

> An arithmetic series is the indicated *sum* of an arithmetic sequence.

S_{10} denotes the sum of the arithmetic series to 10 terms.

Let us observe a familiar method of finding the sum of an arithmetic series.

$$S_5 = 3 + 5 + 7 + 9 + 11$$
$$S_5 = 11 + 9 + 7 + 5 + 3 \quad (\textit{Note reverse order!})$$

Add $2S_5 = 14 + 14 + 14 + 14 + 14$ (*Note!* The sum of t_1 and $t_5 = 14$.)

$$2S_5 = 5 \times 14$$
$$S_5 = \frac{5 \times 14}{2}$$
$$S_5 = 35$$

Similarly

$$S_{10} = 3 + 5 + 7 + 9 + \ldots + 17 + 19 + 21$$
$$S_{10} = 21 + 19 + 17 + \ldots + 7 + 5 + 3$$

Add $2S_{10} = 24 + 24 + 24 + \ldots + 24 + 24 + 24$

$$2S_{10} = 10 \times 24$$
$$S_{10} = \frac{10 \times 24}{2}$$
$$S_{10} = 120$$

To find the sum of n terms of the general arithmetic series:

$$S_n = t_1 + (t_1 + d) + (t_1 + 2d) + \ldots + (t_n - 2d) + (t_n - d) + t_n$$
$$S_n = t_n + (t_n - d) + (t_n - 2d) + \ldots + (t_1 + 2d) + (t_1 + d) + t_1$$
$$2S_n = (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \ldots + (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n)$$
$$= n(t_1 + t_n)$$

(1) $$\therefore S_n = \frac{n}{2}(t_1 + t_n)$$

But $$t_n = t_1 + (n-1)d$$

By substitution into formula (1) for S_n we obtain an alternative formula.

$$S_n = \frac{n}{2}(t_1 + t_n)$$
$$= \frac{n}{2}\left[t_1 + t_1 + (n-1)d\right]$$

(2) $$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

In formula (1) above, the variables are S_n, n, t_1, and t_n.

In formula (2) above, the variables are S_n, n, t_1 and d.

Therefore, for either formula, if any three variables are given, the corresponding fourth variable may be found.

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

Example 1

Find the sum of the series

$$6 + 13 + 20 + \ldots$$

to 12 terms.

Solution

First, list the variables which are given:

$$t_1 = 6,\ d = 7,\ n = 12.$$

These indicate that we should choose formula (2).

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$
$$S_{12} = \frac{12}{2}\left[2(6) + (12-1)7\right]$$
$$= 6\,[12 + 77]$$
$$= 534$$

Example 2

Find the sum of the 16 terms of an arithmetic series given the first term is −15 and the last term 81.

Solution

The given variables are

$$t_1 = 15,\ n = 16,\ t_{16} = 81.$$

We note formula (1) is more suitable for the solution. Why?

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_{16} = \frac{16}{2}(-15 + 81)$$
$$= 8\,(66)$$
$$= 528$$

Example 3

Find the number of terms of the series

$$16 + 12 + 8 + \ldots$$

which will give a sum of 36.

Solution

Given $t_1 = 16,\ d = -4,\ S_n = 36.$

$$S_n = \frac{n}{2}[2t_1 + (n-1)\,d]$$

$$36 = \frac{n}{2}[2(16) + (n-1)(-4)]$$

$$36 = \frac{n}{2}[32 - 4n + 4]$$

$$36 = \frac{n}{2}[36 - 4n]$$

$$36 = 18n - 2n^2$$
$$n^2 - 9n + 18 = 0$$
$$(n-6)(n-3) = 0$$
$$n - 6 = 0 \text{ or } n - 3 = 0$$
$$n = 6 \text{ or } n = 3$$

The number of terms is either 3 or 6. Why can there be two different values for n?

Example 4

Find the sum of the series

$$7 + 10 + 13 + 16 + \ldots + 70.$$

Solution

First, list the variables which are given.

$$t_1 = 7,\ d = 3,\ t_n = 70.$$

This is insufficient for either formula. (Why?) We must solve for n first, using the general term.

$$t_n = t_1 + (n-1)\,d$$
$$70 = 7 + (n-1)\,3$$
$$70 = 7 + 3n - 3$$
$$66 = 3n \rightarrow n = 22$$

Now we have sufficient given variables to use either sum formula.

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_{22} = \frac{22}{2}(7 + 70)$$
$$= 11\,(77)$$
$$= 847$$

By applying formula (2) the result should be the same. Try it.

Example 5

How many terms of the series

$$25 + 19 + 13 + 7 + 1 + \ldots$$

will yield a sum of 64?

Solution

The given variables are $t_1 = 25,\ d = -6,\ S_n = 64.$

To solve for n use $S_n = \frac{n}{2}[2t_1 + (n-1)\,d]$

$$64 = \frac{n}{2}[2(25) + (n-1)(-6)]$$

$$64 = \frac{n}{2}[50 - 6n + 6]$$

$$64 = 28n - 3n^2$$
$$3n^2 - 28n + 64 = 0$$
$$(3n - 16)(n - 4) = 0$$
$$3n - 16 = 0 \text{ or } n - 4 = 0$$
$$n = \tfrac{16}{3} \text{ or } n = 4$$

$n = 4$ is the correct solution as $n \in N$. (n cannot be a fraction such as $\frac{16}{3}$.)

EXERCISE 3·5

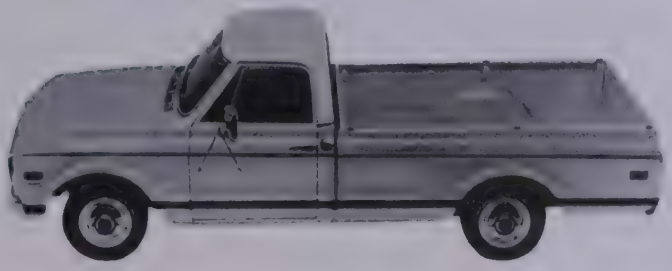
1. Find the sum of each of the following series by using the sum formula.
   (a) $1+3+5+\ldots$ to 9 terms
   (b) $4+6+8+\ldots$ to 8 terms
   (c) $-5+(-2)+1+\ldots$ to 8 terms
   (d) $12+10+8+\ldots$ to 11 terms
   Check your answers for each of the above by long addition.

2. Find the sum of each of the following series:
   (a) first term is 2, last term is 28, number of terms is 10
   (b) $t_1=3,\ t_n=24,\ n=8$
   (c) $t_1=5,\ t_7=29,\ n=7$.

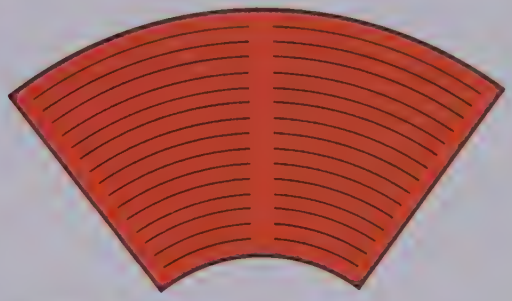
3. Find the sum of each of the following series. (First find the number of terms in each, then apply to the sum formula.)
   (a) $2+5+8+11+\ldots+23$
   (b) $(-15)+(-12)+(-9)+\ldots+18$
   (c) $26+17+8+\ldots+(-19)$

4.

   Roy used reflecting adhesive tape to make the above design on the tailgate of his father's truck. Roy wanted to figure out how much tape he used. He measured the top band at $1\frac{1}{2}$ inches and the bottom band at $15\frac{1}{2}$ inches. Then he counted the number of bands he used for the design. How many inches of tape did Roy use? (Assume the increase in length of each strip from the top down is a constant.)



5. While visiting Greece with her parents, June observed several things relating to Greek theatre. June wished to calculate the total seating of one amphitheatre they visited. She counted the seats in the first four rows and noted they formed a sequence of 30, 34, 38, 42 seats. Then she counted 18 rows and saw that the seating was a regular fan-out as in the diagram. How many seats should June have calculated for the amphitheatre?



6. It is reported that the school master Carl Frederick Gauss had at the age of ten delighted in setting his class to work on problems of this sort:

   Find the sum of these numbers:
   $7285+7323+7361+7399+\ldots+10{,}059$.

   It is further reported Gauss barely allowed his school master to finish before he handed in the answer. Obviously Gauss had not added the numbers. How do you think he got the correct answer? (Incidentally, all the students' answers were wrong with the exception of Gauss's.)

7. Find the sum of each of the following series.
   (a) $4 + 7 + 10 + \ldots + 64$
   (b) $(-19) + (-13) + (-7) + \ldots + 71$
   (c) $42 + 30 + 18 + \ldots + (-258)$

8. How many terms of the series
   (a) $2 + 5 + 8 + 11 + \ldots$ will yield a sum of 155?
   (b) $23 + 18 + 13 + 8 + \ldots$ will yield a sum of -54?

9. Find the sum of the odd numbers between 100 and 200.

10. Find the sum of all numbers ending with the digit 7 between 500 and 1000.

11. The sum is \$630 for the series
    $\$5.00 + \$7.50 + \$10.00 + \ldots$
    How many terms are in the series?

12. If $S_8 = 152$ and $S_{12} = 324$ of an arithmetic series, find the first three terms of this series.

13. From a bridge over a deep gorge Bill timed the drop of a large rock till it hit the river below. His stopwatch recorded the time of fall at 9 seconds. Bill remembered from his studies that an object in free fall drops 16 feet in the first second, 48 feet and 80 feet in the second and third seconds respectively, and so on. What is the height of the bridge above the river below based on Bill's experiment? (Disregard air resistance.)

3·6 Geometric Sequences

Observe the following sequences:
(a) 5, 10, 20, 40, 80, ...
(b) 3, 9, 27, 81, 243, ...
(c) 16, 8, 4, 2, 1, $\frac{1}{2}$, ...
(d) -3, 6, -12, 24, -48, ...
(e) -1, $+1$, -1, $+1$, -1, ...

What is common to all of them?

In what way do these sequences differ from arithmetic sequences? Write the next two terms for each sequence.

A *geometric* sequence is formed by *multiplying* the preceding term by a constant to obtain the next term. The constant in a geometric sequence is called the *common ratio* (r). The common ratio can be determined by dividing any term (except t_1) of the sequence by the term immediately preceding it.

Example 1

For the geometric sequences
(a) 1, 2, 4, 8, 16, ...
(b) -2, -6, -18, -54, ...
determine the common ratios.

Solution

(a) $r = \frac{t_2}{t_1} = \frac{2}{1} = 2$ or $r = \frac{t_5}{t_4} = \frac{16}{8} = 2$

(b) $r = \frac{t_3}{t_2} = \frac{-18}{-6} = 3$ or $r = \frac{t_4}{t_3} = \frac{-54}{-18} = 3$

Have you ever received a chain letter? A simple version of the letter reads like this.

Dear George

Do you want to make a fortune of over $15,000?
I'm certain you do!
All you have to do is copy this letter five times deleting the name at the top of the list and adding your own at the bottom. Send these 5 letters to your friends with the same instructions.
Oh yes, you are to send $5.00 to the person whose name appears first on the list. Each person you send this to will send 5 letters to other friends, and so on. Soon your name will appear on the top of the list and over 3000 people will each send you $5.00 if the chain remains unbroken.
Just imagine over $15,000!
So don't break this chain.
Copy it and mail your $5.00 today to

J.M. Doped 35 Maple Drive
Paradise Creek
W.E. Dull 5014 - 6 St. S.W.
Smalltown
W.R. Slough 36 Barnaby Ave.
Sunshine Village
C.M. Gyped 17042 Brooklyn Bridge
Watertown
R.U. Smarth 52 - 16A Street
Hicksville

Chain letters and pyramids crop up here

Chain letters and their corporate equivalent, pyramid-selling operations, happen regularly, he says, but not often.

As well, says Det. - Sgt. Askew, the squad has had one conviction for chain-letter selling. The letter would have been worth $8,000 if the chain had not been broken, he says.

Chain-letter operators are easy to convict, he says, because people at the end of the chain are eager to testify. "The only people who derive anything are the people who start the chain letter."

"We try to keep a pretty close eye on it," says Det.-Sgt. Askew.

For discussion:

(a) If the chain is unbroken, how many people will send R.U. Smarth $5.00?

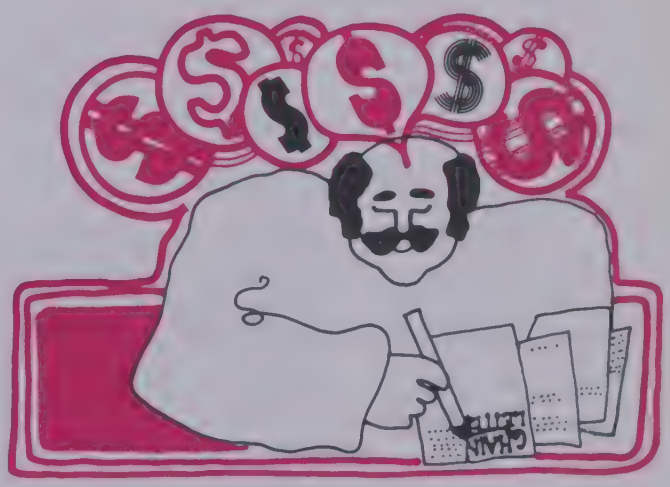
(b) Would you break the chain or would you do as the letter suggests? Why?

(c) Discuss the types of chain letters members of your class have received or seen.

(d) Why do you think people actually "fall" for the chain letter gimmick?

(e) Discuss why there would be a law against chain letters.

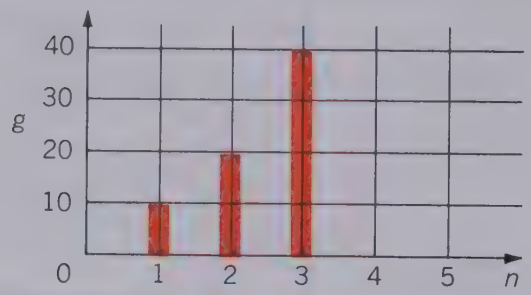
(f) Ask a representative of the police department to discuss the law with reference to chain letters and "pyramid sales firms". What is meant by pyramid sales firms? Why do you think there are laws prohibiting pyramid sales firms?



EXERCISE 3·6

1. For each of the sequences at the beginning of Section 3·6, determine the common ratio and then find the next four terms.

2. Which patterns in Section 3·1 are geometric sequences? What is the common ratio of each of those sequences?

3. 

What should the next two bars on this graph read?

   (a) Label a section of squared paper similar to the sample above. By means of a bar graph illustrate the following geometric sequence:
   5, 15, 45, 135, 405.

   (b) On another section, graph the sequence
   $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$
   using point-dots.

   (c) Discuss the difference in graphs of geometric sequences and arithmetic sequences.

3·7 The General Term of Geometric Sequences

Consider the geometric sequence
1, 3, 9, 27, ...
with $t_1 = 1$ and $r = 3$.

$t_2 = 1 \times 3 = 3$ i.e. $t_2 = t_1 \times r = t_1 r$

$t_3 = 3 \times 3$ i.e. $t_3 = t_1 r \times r$
$= 1 \times 3^2$ $= t_1 \times r^2$
$= 9$ $= t_1 r^2$

$t_4 = 9 \times 3$ i.e. $t_4 = t_1 r^2 \times r$
$= (1 \times 3^2) \times 3$ $= t_1 r^3$
$= 1 \times 3^3$

In general form a geometric sequence is
$t_1, t_1 r, t_1 r^2, t_1 r^3, \ldots, t_1 r^{n-1}, \ldots$

where $\boxed{t_n = t_1 r^{n-1}}$

This is called the *nth term* or the *general term* of a geometric sequence.

Example 1

Find the sixth term of the series,
2, 6, 18, ...

Solution

$t_1 = 2, r = 3, n = 6$
$t_n = t_1 r^{n-1}$
$t_6 = 2\,(3)^{6-1}$
$= 2\,(3)^5$
$= 2\,(243)$
$= 486$

Example 2

Which term of the sequence
$-81, 27, -9, \ldots$
is $\frac{1}{27}$?

Solution

$t_1 = -81, r = -\frac{1}{3}, t_n = \frac{1}{27}$, find n.

$t_n = t_1 r^{n-1}$
$\frac{1}{27} = -81(-\frac{1}{3})^{n-1}$
$\frac{1}{27(-81)} = (-\frac{1}{3})^{n-1}$
$\frac{1}{-(3)^3(3)^4} = (-\frac{1}{3})^{n-1}$
$(-\frac{1}{3})^7 = (-\frac{1}{3})^{n-1}$
$\therefore n - 1 = 7, n = 8$

$\frac{1}{27}$ is the eighth term of the sequence. This can be checked by extending the sequence until the term $\frac{1}{27}$ is reached.

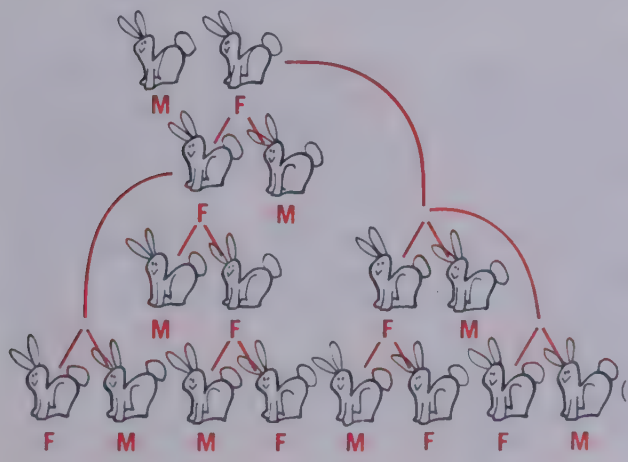
$$t^n = t_1 r^{n-1}$$

EXERCISE 3·7

1. Find the next two terms of each of these sequences.
   (a) 1, 4, 16, ...
   (b) 5, 15, 45, ...
   (c) 3, −6, 12, ...
   (d) 160, 80, 40, ...
   (e) −27, −9, −3, ...
   (f) 0.002, 0.008, 0.032, ...
   (g) $\frac{1}{81}, \frac{1}{9}, 1, \ldots$

2. What is the ratio in each of the sequences?
   (a) 12, −36, 108, −324, ...
   (b) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$
   (c) 0.002, 0.010, 0.050, 0.250, ...
   (d) 105, 2100, 42,000, ...
   (e) −4, 10, −25, ...

3. Bobby studied the breeding of rabbits under controlled conditions. He began with a pair, one male and one female. After four months the pair produced two young ones (one m., one f.). Bobby separated all of them until the young ones matured. Then he allowed them together and each female produced a pair (one m., one f.). Again he separated them until the young matured. After mating, the third set of births were pairs (one m., one f.) for each female, all as illustrated below. A new set of births occurred every four months. If this pattern continued for 2 full years, how many rabbits would be born in the last set?



4. For the sequence
   (a) 5, 10, 20, 40, ... find the 7th term.
   (b) −3, −6, −12, ... find the 9th term.
   (c) $\frac{81}{8}, \frac{27}{4}, \frac{9}{2}, \ldots$ find the 8th term.
   (d) $\frac{1}{256}, \frac{1}{64}, \frac{1}{16}, \ldots$ find the 7th term.

5. Which term of the sequence
   (a) 4, 12, 36, ... is 972?
   (b) $\frac{1}{256}, \frac{1}{64}, \frac{1}{16}, \ldots$ is 4096?
   (c) 128, 192, 288, ... is 2187?

3·8 Geometric Series and the Sum of Geometric Series

A *geometric series* is the indicated *sum* of a geometric sequence.

Thus, $1 + 2 + 4 + 8 + 16 + \ldots$
is an example of a geometric series.

$$S_6 = 1 + 2 + 4 + 8 + 16 + 32$$
$$= 63$$

S_6 denotes the *sum of the geometric series* to 6 terms. Note this method of finding the sum of a geometric series.

$$S_6 = 1 + 2 + 4 + 8 + 16 + 32 \quad ①$$

① × 2 $\quad 2S_6 = \quad 2 + 4 + 8 + 16 + 32 + 64 \quad ②$

② − ① $\quad S_6 = 64 - 1$ *Multiply by 2 because it*
$\quad = 63$ *is the common ratio.*

Consider another example.

$$S_5 = 2 + 6 + 18 + 54 + 162 \quad ①$$

① × 3 $\quad 3S_5 = \quad 6 + 18 + 54 + 162 + 486 \quad ②$

② − ① $\quad 2S_5 = 486 - 2$ *Why multiply by 3?*

÷ $\quad S_5 = \frac{484}{2}$

$$S_5 = 242$$

Now let us find the sum of the geometric series in general form.

$$S_n = t_1 + t_1 r + t_1 r^2 + \ldots + t_1 r^{n-1}$$

① × r $\quad rS_n = \quad t_1 r + t_1 r^2 + \ldots + t_1 r^{n-1} + t_1 r^n$

② − ① $rS_n - S_n = t_1 r^n - t_1$ *Why multiply by r?*

$$S_n(r-1) = t_1(r^n - 1)$$

$$S_n = \frac{t_1(r^n - 1)}{r - 1} \quad (r \neq 1)$$

This formula is used to find the sum of a geometric series when the given variables are t_1, r and n.

Example 1

Find the sum of the series

$$2, 6, 18, \ldots$$

to 6 terms.

Solution

$$t_1 = 2, r = 3, n = 6$$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_6 = \frac{2(3^6 - 1)}{3 - 1}$$

$$= \frac{2(729 - 1)}{2}$$

$$= 728$$

Example 2

Find the sum of the geometric series

$$6 + 12 + 24 + \ldots + 1536.$$

Solution

$$t_1 = 6, r = 2, t_n = 1536$$

We must first find n:

$$t_n = t_1 r^{n-1}$$

$$1536 = 6(2)^{n-1}$$

$$256 = 2^{n-1}$$

$$2^8 = 2^{n-1}$$

$$n - 1 = 8 \rightarrow n = 9$$

Now use the sum formula:

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_9 = \frac{6(2^9 - 1)}{2 - 1}$$

$$= \frac{6(512 - 1)}{1}$$

$$= 6(511)$$

$$= 3066$$

Example 3

Find the sum of the series

$$(-1) + 3 + (-9) + 27 + \ldots$$

to 8 terms.

Solution

$$t_1 = -1, r = -3, n = 8$$

Use the sum formula:

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$= \frac{(-1)[(-3)^8 - 1]}{-3 - 1}$$

$$= \frac{(-1)(6561 - 1)}{-4}$$

$$= \frac{6560}{4}$$

$$= 1640$$

EXERCISE 3·8

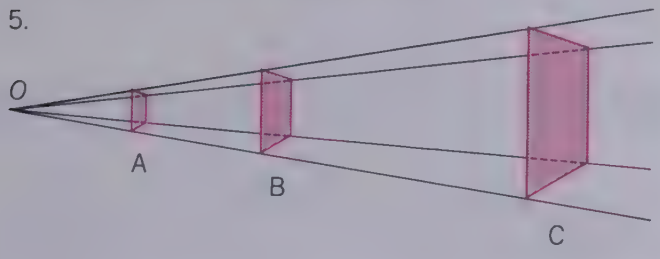
1. Find the sum of the first 7 terms of each series.
   (a) $(-1) + (1) + (-1) + (1) + \ldots$
   (b) $(-8) + (-4) + (-2) + \ldots$
   (c) $3 + 6 + 12 + \ldots$
   (d) $64 + 32 + 16 + \ldots$

2. Certain animal cells divide into two every $3\frac{1}{2}$ hours.
   (a) If a cell culture began with one cell, how many cells might one expect after 28 hours of cell division? How many cells would there be after 27 hours?
   (b) If a cell culture began with 10 cells, how many cells might one expect after 35 hours of cell division?

3. Find the sum of each of these series.
   (a) $112 + 56 + 28 + \ldots$ to 5 terms
   (b) $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \ldots$ to 7 terms
   (c) $18 + 6 + 2 + \ldots$ to 7 terms
   (d) $256 + 128 + 64 + \ldots$ to 9 terms
   (e) $1 + \frac{4}{3} + \frac{16}{9} + \ldots$ to 5 terms.

4. Refer to question 5 in Exercise 3·7. How many rabbits in all would Bobby have if the pattern of births continued for 32 months?

5.



For a project in the laboratory Rick and Jean set up the above model of string and cardboard. Piece A is a 1-cm square and is 3 cm from *O*. Piece B is a 2-cm square and is 6 cm from *O*. Piece C is a 4-cm square and is 12 cm from *O*. They set up piece D at 24 cm from *O*, and piece E at 48 cm from *O*. They discovered that the size of the squares A, B, C, D, E formed a geometric sequence when taken in that order.

(a) Find the total area of the cardboard squares which they used.

(b) If they had continued the model to the seventh piece, how large in area would that piece be?

6. A fan-out system of communication is used by the students' union of a large high school. The president and secretary each phone four members to make an urgent announcement. Each member contacted then phones four other members, who each in turn phone four again. After the sixth full wave of phoning one student is left to be called. How many students make up the union?

7. If you were offered a job to last 20 days, for which the wages paid would be 1¢ for the first day, 2¢ for the second day, 4¢ for the third day, 8¢ for the fourth, etc., would you consider taking the job? How much would be the wages for the 20 days?

8. If $t_2 = 6$ and $t_4 = 108$ of a geometric series, find the sum of this series to 6 terms.

9. For a geometric series $S_3 = 1\frac{3}{4}$ and $S_6 = 15\frac{3}{4}$. Find the sum of this series to 10 terms.

10. The sum of the second and third terms of a geometric series is 48 and of the fourth and fifth terms is 432. Find the sum of this series to 5 terms.

Using the Library

1. Research these names and events and place them with an appropriate comment on your time line:
   - Late Stone Age
   - David becomes King of Israel
   - Discovery of copper or metals (approximate)
   - Stonehenge built
   - Pyramids built – Great Pyramid of Gizeh
   - Hammurabi
   - Moscow papyrus
   - Rhind papyrus
   - Ramesses II
   - Fall of Nineveh
   - Thales
   - Pythagoras
   - Nebuchadnezzar
2. What is thought to be the origin of the rhyme:
   As I was going to St. Ives, al...
   What are some of the earlier versions?
   An Introductory History to Mathematics – Eves
3. The Egyptians had a method of multiplication similar to that known now as duplation and mediation. Research this method then multiply 56 × 77.
4. Trace the history of number mysticism in the East.
5. Discuss why the period indicated in question 1 was lacking in power to advance mathematically or scientifically.

REVIEW EXERCISE 3

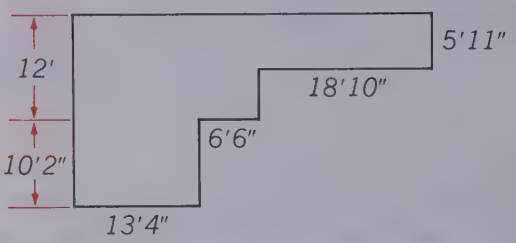
1. For the sequence: $-23, -19, -15, -11, \ldots$
   (a) find the 41st term
   (b) find the sum of the first 13 terms.
2. Write the first five terms of an arithmetic sequence for which $t_{31} = 33$ and $t_{25} = 21$.
3. Find the sum of the first 9 terms of a geometric series for which $t_3 = 9$ and $t_7 = \frac{1}{9}$.

Your Job and Mathematics

The Walk-In-Comfort Carpet Co.

In the retail carpet business mathematics and common sense are used by every employee. The three problems given here illustrate how the cashier, salesman, and accountant might use mathematics.

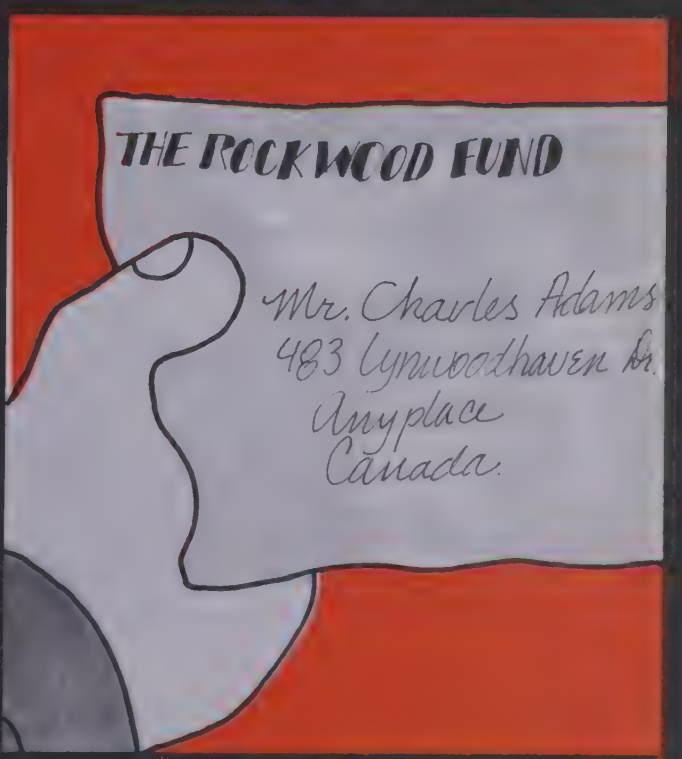
1. Mrs. A. Brodlume, a carpet saleswoman, has been asked to give Mr. and Mrs. Suburbia a quotation on wall-to-wall shag rug at $11.95 a yard (this means per square yard) for the floor plan shown.



   The rug comes in rolls of 12-foot widths. The customer pays for all waste. The customer will want the minimum amount of joining or seams. How much would you charge for the shag rug? Compare your answer with your friend's. Show how you would plan the rug and the seams.
2. The accountant must calculate the true interest rate on time-payment purchases made by the company. For example, The Walk-In-Comfort Carpet Co. purchases 1500 yards of carpet at $9.87 a yard. The company receives the bill requesting an initial payment at time of delivery of $3400, then $3600 in 30 days, $3600 in 60 days, and $5000 in 90 days. Is the true interest rate reasonable? What constitutes a reasonable interest rate? Quote some current interest rates listed by newspapers, banks, etc. as support for your answers.
3. The cashier notes that there is a sale on oriental $8′ \times 12′$ rugs. The rug listed at $124 is marked at 20% off. As an employee the cashier is entitled to a further 10% discount. There is a 4% sales tax. What is the price of the rug for
   (a) a customer?
   (b) an employee like herself?

Chapter Four

ANNUITIES



4·1 Simple Interest (Review)

Example 1

Find the simple interest on
(a) \$300 at 8% per annum for 2 years.
(b) m dollars at n% per annum for p years.

Solution

(a) Principal (p) = \$300
Rate of interest (r) = 8% = $\frac{8}{100}$
Time (t) = 2 years
Interest (I)$_8$ = prt
$I = \$300 \times \frac{8}{100} \times 2 = \$48$

(b) $p = \$m$, $r = n\% = \frac{n}{100}$, $t = p$
$I = prt$
$I = \$m \times \frac{n}{100} \times p = \frac{mnp}{100}$ dollars

EXERCISE 4·1

1. Find the interest on:
   (a) \$700 at 5% per annum for 2 years
   (b) \$1850 at 6% per annum for $1\frac{1}{2}$ years
   (c) \$1200 at 7% per annum for 5 months.

2. James Roberts inherited \$2000. He deposited it into a nonchequing savings account which pays 8% per annum.
   Calculate:
   (a) The interest earned by this account in one year.
   (b) The amount in James' account at the end of one year.
   (c) The principal of his account for the second year.
   (d) The interest paid to the account after the second year.
   (e) The amount in James' account at the end of two years.
   (f) The interest earned at the end of the third year.
   (g) The amount after three years.

   Notice that James' account was increasing by two methods:
   (i) interest on the principal,
   (ii) interest on interest.

   When we calculate interest this way it is known as *compound interest.*

3. (a) What is the simple interest on \$2000 at 8% per annum for three years?
   (b) Take the interest from (a) and add it to \$2000. What is the amount?
   (c) How does this amount compare with the amount from 2(g)?

4. Lisa White deposited \$600 into an account which pays 8% per annum semiannually.
   (a) How many times a year is the interest calculated?
   (b) Find the interest earned by Lisa's account at the end of $\frac{1}{2}$ year.
   Hint! $p = \$600$, $r = \frac{8}{100}$, $t = \frac{1}{2}$
   (c) What is the interest on \$600 at 4%?
   (d) Compare parts (b) and (c). What do you observe? Write a general statement of your observation.
   (e) What is the amount of Lisa's account after the first $\frac{1}{2}$ year?
   (f) What is the amount of her account after one year? after $1\frac{1}{2}$ years?

5. If the interest rate is 6% per annum compounded semiannually, then *the rate (i) for one interest period* is .03 or $\frac{3}{100}$.
   (a) If interest rate is 10% per annum compounded semiannually, find i.
   (b) If interest rate is 6% per annum compounded quarterly (every three months), find i.
   (c) If interest rate is 5% per annum compounded every six months, find i.

4·2 Compound Interest

Let us recall the formula for calculating the amount (A) if the principal (P), the compound interest rate per period (i) and the number of interest periods (payments) (n) were given.

$A = P + Pi$
$= P(1 + i)$ after 1 interest period.

$A = P(1 + i) + P(1 + i)i$
$= P(1 + i)(1 + i)$ *common factor*
$= P(1 + i)^2$ after 2 interest payments.

$A = P(1 + i)^2 + P(1 + i)^2 i$
$= P(1 + i)^2(1 + i)$ *common factor*
$= P(1 + i)^3$ after 3 interest payments

$A = P(1 + i)^3 + P(1 + i)^3 i$
$= P(1 + i)^3(1 + i)$ *common factor*
$= P(1 + i)^4$ after 4 interest payments
and

$\boxed{A = P(1 + i)^n}$ after n interest payments.

How to Use Amount Tables

COMPOUND AMOUNT OF 1

Yrs.	4%	4½%	5%	6%	7%	8%	Yrs.
1	1.04000	1.04500	1.05000	1.06000	1.07000	1.08000	1
2	1.08160	1.09203	1.10250	1.12360	1.14490	1.16640	2
3	1.12486	1.14117	1.15763	1.19102	1.22504	1.25971	3
4	1.16986	1.19252	1.21551	1.26248	1.31080	1.36049	4
5	1.21665	1.24618	1.27628	1.33823	1.40255	1.46933	5
6	1.26532	1.30226	1.34010	1.41852	1.50073	1.58687	6
7	1.31593	1.36086	1.40710	1.50363	1.60578	1.71382	7
8	1.36857	1.42210	1.47746	1.59385	1.71819	1.85093	8
9	1.42331	1.48610	1.55133	1.68948	1.83846	1.99900	9
10	1.48024	1.55297	1.62889	1.79085	1.96715	2.15893	10
11	1.53945	1.62285	1.71034	1.89830	2.10485	2.33164	11
12	1.60103	1.69588	1.79586	2.01220	2.25219	2.51817	12
13	1.66507	1.77220	1.88565	2.13293	2.40985	2.71962	13
14	1.73168	1.85194	1.97993	2.26090	2.57853	2.93719	14
15	1.80094	1.93528	2.07893	2.39656	2.75903	3.17217	15

When $i = .05$ and $n = 4$ (see Example 1)
then $(1.05)^4 = 1.21551$ (at 5% for 4 years).

When $i = .04$ and $n = 10$ (see Example 2)
then $(1.04)^{10} = 1.48024$.

When $i = .07$ and $n = 12$
then $(1.07)^{12} = 2.25219$.

Example 1

Find the amount of $2000 invested at 5% per annum compounded annually for 4 years.

Solution

$P = \$2000$, $i = 5\% = .05$, $n = 4$

Using the formula:

$A = P(1 + i)^n$
$= 2000(1 + .05)^4$
$= 2000(1.05)^4$
$= 2000(1.21551)$ *from Amount of 1 Tables*
$= \$2431.02$

Example 2

At age 14 John banked $400 to save for a trip to Europe when he was 19. Find the amount of $400 invested at 8% annually compounded semiannually for 5 years.

Solution

Because interest is paid every $\frac{1}{2}$ year at 8% annually for 5 years, then $i = .04$ and $n = 10$ (number of interest payments).

$A = P(1 + i)^n$
$= 400(1 + .04)^{10}$
$= 400(1.04)^{10}$
$= 400(1.48024)$ *from Amount of 1 Tables*
$= \$592.10$

The amount is $592.10.

Example 3

If $1000 was invested at 6% annually compounded quarterly, what would it amount to in 6 years?

Solution

$P = \$1000, n = 24, i = 1\frac{1}{2}\% = .015$

Note!

$$i = \frac{\text{annual rate}}{\text{no. of interest periods per year}}$$

n = (no. of years) × (no. of interest periods/year).

$A = P(1 + i)^n$
$= 1000(1.015)^{24}$
$= 1000(1.42950)$ *from Amount of 1 Tables*
$= \$1429.50$

The amount is $1429.50.

EXERCISE 4·2

1. State what i and n are in each of the following:
   (a) $100 invested at 6% per annum compounded annually for 5 years.
   (b) $800 invested at 4% per annum compounded semiannually for 12 years.
   (c) $1200 invested at 8% per annum compounded quarterly for 5 years.

2. Find the amount of:
   (a) $100 invested at 6% per annum compounded annually for 5 years.
   (b) $800 invested at 4% per annum compounded annually for 12 years.

3. If $1200 is invested at 8% per annum compounded semiannually what will the amount be after 6 years?

4. What is the amount of $1000 invested at 6% per annum compounded semiannually for 6 years?

5. Fred deposited $600 into a nonchequing account at 7% per annum compounded semiannually for 4 years. What is the amount at the end of 4 years?

6. Mr. White invested $2000 at 8% per annum compounded quarterly for 5 years. What was the amount of this investment after 5 years?

4·3 Present Value

In Section 4·2 we calculated the amount to which a lump sum (principal) investment accrued after several periods (n) of compound interest. In this section we shall calculate what the *lump sum (principal)* must be in order that a certain amount (A) be obtained after a number of periods (n) of compound interest. The following example illustrates a typical problem. Note that the lump sum (principal) is now called the *Present Value*. Note also that Present Value tables simplify the computation.

Example 1

How much must one invest now at 7% per annum compounded annually in order that the amount be $10,000 after 6 years?

Solution 1

$A = \$10{,}000, i = .07, n = 6$, find P.

$$A = P(1 + i)^n$$
$$\$10{,}000 = P(1 + .07)^6$$
$$\frac{\$10{,}000}{(1.07)^6} = P \quad \textit{from Amount of 1 Tables}$$
$$P = \frac{\$10{,}000}{1.50073} = \$6663.42$$

This amount is called the *Present Value.*

Solution 2

$A = \$10{,}000, i = .07, n = 6$, find P.

$A = P(1 + i)^n$
$P = A(1 + i)^{-n}$ *How do we derive this form?*
$= 10{,}000(1 + .07)^{-6}$
$= 10{,}000(1.07)^{-6}$
$= 10{,}000(0.66634)$ *from Present Value of 1 Tables*
$= \$6663.40.$

$$P = A(1 + i)^{-n}$$

How to Use Present Value Tables

PRESENT VALUE OF 1

Yrs.	4%	4½%	5%	6%	7%	8%	Yrs.
6	0.79031	0.76790	0.74622	0.70496	0.66634	0.63017	6
7	0.75992	0.73483	0.71068	0.66506	0.62275	0.58349	7
8	0.73069	0.70319	0.67684	0.62741	0.58201	0.54027	8
9	0.70259	0.67290	0.64461	0.59190	0.54393	0.50025	9
10	0.67556	0.64393	0.61391	0.55840	0.50835	0.46319	10
11	0.64958	0.61620	0.58468	0.52679	0.47509	0.42888	11
12	0.62460	0.58966	0.55684	0.49697	0.44401	0.39711	12
13	0.60057	0.56427	0.53032	0.46884	0.41496	0.36770	13
14	0.57748	0.53997	0.50507	0.44230	0.38782	0.34046	14
15	0.55526	0.51672	0.48102	0.41727	0.36245	0.31524	15

When $i = .07$ and $n = 6$ *from Example 1, Section 4·3* then $(1.07)^{-6} = .66634$.

When $i = .05$ and $n = 12$, then $(1.05)^{-12} = .55684$.

EXERCISE 4·3

1. How do the Amount of 1 Tables relate to the Present Value of 1 Tables?

2. Use Present Value of 1 Tables to find the values of:

   (i) $(1.06)^{-7}$ (ii) $(1.04)^{-12}$ (iii) $(1.08)^{-5}$

   (iv) $(1.045)^{-16}$ (v) $(1.05)^{-20}$ (vi) $(1.015)^{-18}$

3. Find the present value of a loan that amounts to $3000 after 2 years compounded semiannually at 6% per annum.

4. Find the present value of each loan:

	Amount	Time	Annual interest rate	Compounded
(a)	$4500	6 years	8%	Semiannually
(b)	900	2 years	6%	Annually
(c)	500	3 years	18%	Semiannually

5. Find the sum of money needed to be invested today to accrue $5000 in 3 years at 16% per annum compounded semiannually.

6. Mr. Davis, who turned 50 years old, wishes to add $20,000 to his pension by his 60th birthday. How much will he have to put into a retirement savings plan now at 6% per annum compounded annually to add the above amount at age 60?

7. Jerry's father invested on Jerry's 16th birthday a sum of money at 7% per annum compounded semiannually to make available to Jerry on his 21st birthday the amount of $5000. What sum did his father invest? (This sum is known as the Present Value.)

8. Find the present value of each of the following:

   (a) An amount of $50,000 which accumulated from an investment at 5% per annum compounded semiannually for 20 years.

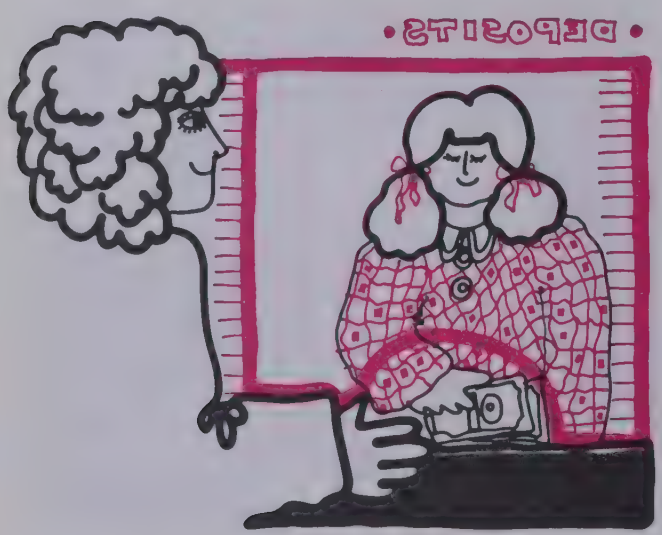
   (b) An amount of $15,000 from an investment at 7% per annum compounded semiannually for 7 years.

4·4 Ordinary Annuities

An *annuity* is a sequence of regular payments made over a period of time.

An *ordinary annuity* is one where the payments are made at the *end* of each time interval.

After Betty completed her eighth grade her Uncle Joe offered her a summer job at his market garden. Betty was able to save $100 that summer which she invested at 6% per annum compounded annually. After her ninth grade she worked again for Uncle Joe and added another $100 to her investment. She was able to do this for four consecutive years. How much did Betty have after her fourth deposit?



The first $100 earned interest for 3 years.
$\therefore A_1 = \$100\,(1.06)^3$

The second $100 earned interest for 2 years.
$\therefore A_2 = \$100\,(1.06)^2$

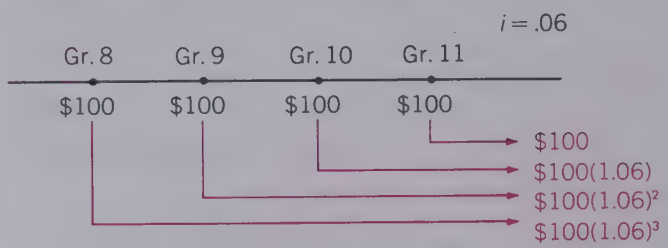
The third $100 earned interest for 1 year.
$\therefore A_3 = \$100\,(1.06)$

The last $100 earned no interest ($A_4$).

Betty's investment amounted to
$A_4 + A_3 + A_2 + A_1$
$= \$100 + \$100\,(1.06) + \$100\,(1.06)^2 + \$100\,(1.06)^3$

But the terms of this amount form a geometric sequence (see Chapter 3) with
$t_1 = \$100$, $r = 1.06$, and $n = 4$.

Let us illustrate this by a time diagram.

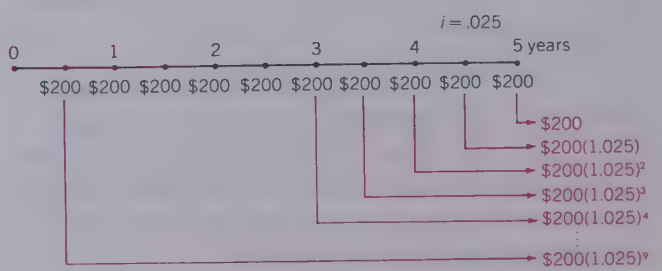


Examples of annuities are: mortgage payments, pensions, retirement savings, budget buying, etc. The situations described in the examples and exercises that follow will illustrate various uses of ordinary annuities.

Example 1

Sally's mother deposits $200 every six months into a retirement savings plan which pays her interest at 5% per annum compounded semiannually. Illustrate by a time diagram the sequence formed for the first five years of deposits.

Solution



EXERCISE 4·4

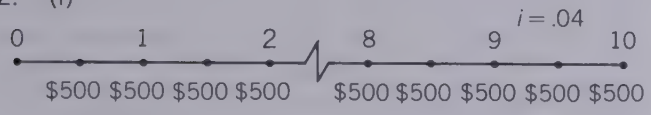
1. Illustrate by a time diagram the sequence formed for each of the following:

   (a) $500 deposited into a trust account paying 5% annually for six years.

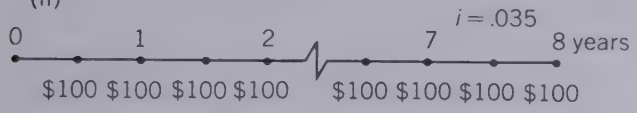
   (b) $600 deposited into a retirement savings plan every six months at 6% per annum compounded semiannually for four years.

   (c) $200 deposited into an account every three months at 6% annually compounded quarterly for 3 years.

2. (i)



(ii)



For each time diagram above:

(a) What is the rate of interest per annum if it is compounded semiannually?

(b) What will be the first and last terms of the sequence?

(c) For each sequence formed what is the common ratio and the number of terms?

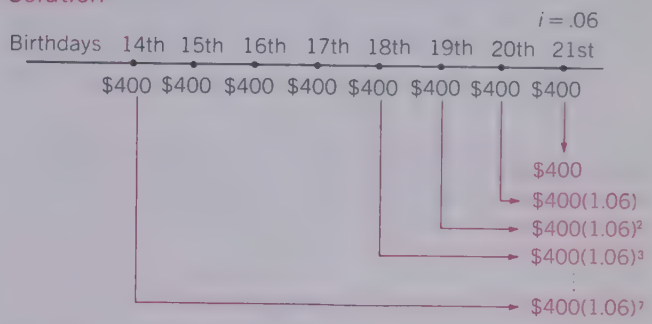
4·5 Amount of an Annuity

Very often it is desirable to know the amount of an annuity at the end of a certain period of time. To find the amount we simply calculate the sum of the geometric series representing the annuity.

Example 1

Jean's father set up an account for her at his bank on her 14th birthday. He deposited $400 at 6% per annum compounded annually. He repeated the deposits on her 15th, 16th and 17th birthdays. He also promised to continue the annual deposits until her 21st birthday. Jean calculated what the amount would be on her 21st birthday. This is how she worked it.

Solution



$$\text{Amount} = \$400 + \$400\,(1.06) + \$400\,(1.06)^2 + \ldots + \$400\,(1.06)^7$$

This forms a geometric series. Therefore the amount is the sum of the geometric series.

$t_1 = \$400,\ r = 1.06,\ n = 8$

$$\text{Amount} = S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$\text{Amount} = \frac{400[(1.06)^8 - 1)]}{1.06 - 1}$$

$$= \frac{400(1.59385 - 1)}{.06} \quad \textit{from Amount of 1 Tables}$$

$$= \frac{400(.59385)}{.06}$$

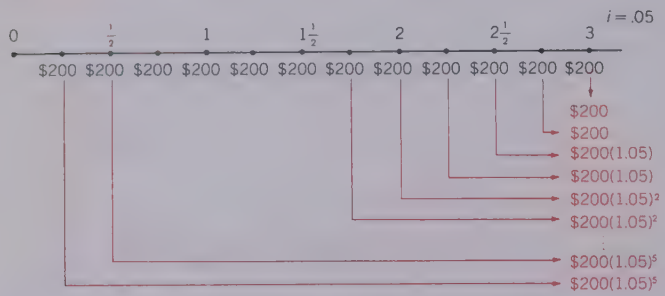
$$= \$3959.00$$

The amount of the annuity is $3959.00.

Example 2

Mr. Thyme invests a sum of money such that he is making deposits of $200 every three months at 10% per annum compounded semiannually. He makes deposits for three years. What amount does he have at the end of three years?

Solution



$$A = \$200 + \$200 + \$200(1.05) + \$200(1.05) + \$200(1.05)^2 + \$200(1.05)^2 + \ldots + \$200(1.05)^5$$
$$= \$400 + \$400(1.05) + \$400(1.05)^2 + \ldots + \$400(1.05)^5$$

$t_1 = \$400,\ r = 1.05,\ n = 6$

$$A = S_n = \frac{t_1(r^n - 1)}{r - 1}$$
$$= \frac{400\,((1.05)^6 - 1)}{1.05 - 1}$$
$$= \frac{400\,(1.35000 - 1)}{.05} \quad \textit{from Amount of 1 Tables}$$
$$= \frac{400\,(.35000)}{.05}$$
$$= \$2800.00$$

Why was there no interest paid on the second last $200 payment?

Why is the interest on the first $200 the same as on the second $200 – after all, the company had the first $200 longer than the second $200?

EXERCISE 4·5

1. (a) Use Amount of 1 Tables to find the values of:
   (i) $(1.05)^6$ (ii) $(1.07)^4$ (iii) $(1.03)^{20}$
   (iv) $(1.08)^{10}$ (v) $(1.025)^{12}$ (vi) $(1.035)^{15}$

2. Consider and draw a time diagram for each of the following amount annuities.
   (a) Annual deposits of $500 at 4% per annum compounded annually for 5 years.
   (b) Half-yearly deposits of $300 at 6% per annum compounded semiannually for 2 years.

   Then, (i) Write the series that represents the amount for each.
   (ii) State the value of t_1, r, n for each.
   (iii) Find the amount of each annuity.

3. Jim and Sue (after they were married) began saving for a down payment on a house. They invested $600 every six months into a plan which paid them 8% per annum compounded semiannually. How much did their savings amount to after $4\frac{1}{2}$ years?

4. Mr. Soakes began with an annual deposit of $1200 into a retirement savings plan on his 41st birthday. The plan paid 7% interest per annum compounded annually. How much is in the savings after the annual deposit on his 60th birthday?

5. Robert was making plans to go into business for himself in four years. He calculated that he could save $200 every three months. If he deposits $200 three months from now and every three months thereafter at 8% per annum compounded semiannually, how much will his savings amount to at the end of 4 years?

6. A manufacturer sets aside $8000 at the end of every year to provide a fund for updating his processes of manufacturing. He invests the money at 7% per annum compounded semiannually. If he makes his first deposit one year from now, how much money will he have in the plan after the eighth deposit?

4·6 Present Value of an Annuity

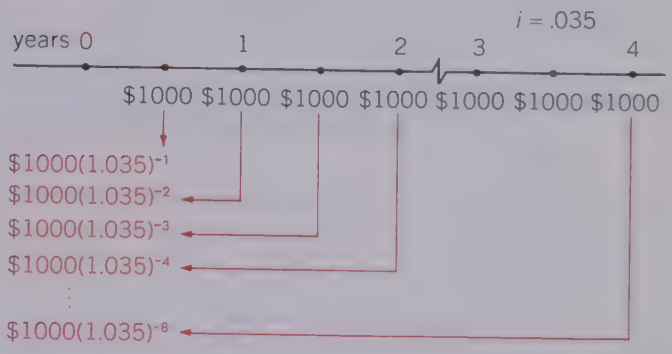
In Section 4·5 we discussed making regular payments, collecting interest on these payments and accumulating a sum of money at some future date. Another annuity problem is illustrated by the case where a sum of money is paid now in order to receive back a given number of payments over a period of years.

For example, a man on retirement may invest some of his savings in a trust fund which will pay him a fixed monthly salary for a given number of years. He will want to know how much (present value) he will have to pay now. Or a man may decide that he wants to borrow some money. The amount is dependent on the amount he feels he can pay regularly over a given period of time. Mr. John, for example, may feel he can pay $400 semiannually for the next 3 years. He wants to know how much he can get now on this pay-back basis.

Example 1

Mr. Whyte arranged to have a bank pay his son, Jack, a sum of $1000 every six months while attending college. To make this possible Mr. Whyte deposited a sum of money at 7% per annum compounded semi-annually. Jack is to receive his first payment 6 months after the deposit. If Jack is to attend college for 4 years what sum of money was deposited for him? (The sum deposited is called the present value of this annuity.)

Solution



From $A = P(1 + r)^n$, the compound interest formula is

$P = \dfrac{A}{(1 + r)^n}$ or $P = A(1 + r)^{-n}$.

This is the formula used to determine the present value of each $1000 paid out to Jack every 6 months.

Thus,

$$\text{Present Value} = \$1000\,(1.035)^{-1} + \$1000\,(1.035)^{-2} + \$1000\,(1.035)^{-3} + \ldots + \$1000\,(1.035)^{-8}$$

If we rewrite this series by reversing the order of the terms we get:

$$P = \$1000\,(1.035)^{-8} + \$1000\,(1.035)^{-7} + \$1000\,(1.035)^{-6} + \ldots + \$1000\,(1.035)^{-1}$$

where $t_1 = 1000\,(1.035)^{-8}$, $r = 1.035$, $n = 8$.

$$\therefore \text{Present Value } (P) = S_n = \frac{t_1\,(r^n - 1)}{r - 1}$$

$$P = \frac{1000\,(1.035)^{-8}\,[(1.035)^8 - 1]}{1.035 - 1}$$

$$= \frac{1000\,[(1.035)^0 - 1.035^{-8}]}{1.035 - 1}$$

from Present Value of 1 Tables

$$= \frac{1000\,(1 - 0.75941)}{0.035}$$

$$= \frac{1000\,(0.24059)}{0.035}$$

$$= \$6874.00$$

The present value of the annuity is $6874.

EXERCISE 4·6

1. Draw a present value time diagram for each. Then
   (i) write the series that represents the present value of each annuity,
   (ii) state the values of t_1, r, n for each,
   (iii) find the present value of each annuity.
   (a) $100 paid at the end of each year for 3 years if it yielded 6% per annum compounded annually.
   (b) $400 paid at the end of each year for 4 years if it yielded 5% per annum compounded annually.
   (c) $1000 paid at the end of each six months for 3 years if it yielded 6% per annum compounded annually.
   (d) $1000 paid at the end of each six months for 3 years if it yielded 6% per annum compounded semiannually.
   (e) Compare the present values of (c) and (d). Is this what you would expect? Explain.

2. A loan company supplied a cash sum to Mr. Grant which he agreed to repay at $800 every six months at 10% per annum compounded semiannually for five years. What cash sum (present value of the annuity) did Mr. Grant receive?

3. Jake bought a second hand car with no down payment. He had to make payments of $200 every 3 months. Rate of interest charged was 16% per annum compounded quarterly. If he had to make payments for 3 years, what is the present value of his total payments?

4. The Order of Round Hatters building committee proposed the following plan to its membership: a building fund set up so that $800 was to be deposited into an investment account every three months. After five years the fund would have enough for the Order to negotiate a mortgage on their Club House. If the rate of interest was to be 8% per annum compounded semiannually, what was the present value of this fund?

5. Upon retirement at 60, Mr. and Mrs. Hanford invested a large part of their savings in an annuity. This annuity paid them $2000 every three months for 10 years. They purchased the annuity at a rate of 6% per annum compounded semiannually. How much of their savings (present value of the annuity) did they invest?

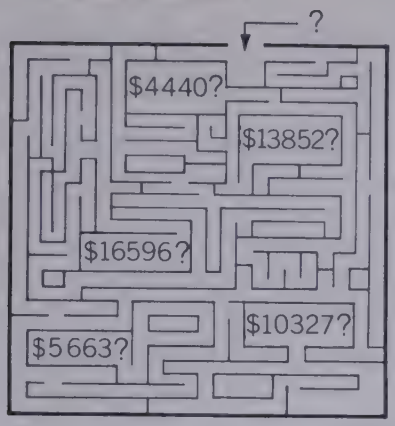
4·7 Periodic Payment of an Ordinary Annuity

We shall consider two cases in this work:

(a) The situation where payments are made regularly to accumulate a certain amount at the end of a fixed period of time. This will be similar to an *Amount of an Annuity* question. See Example 1.

(b) The situation where a certain amount of money is taken now and repaid by fixed payments made regularly over a given period of time. This will be similar to a *Present Value of an Annuity* question. See Example 2.

Invest $10 a month for 444 months. What do you get?



What rate of interest would be necessary to produce the amount indicated? (Use "trial and error" method.)

When working the questions in the exercise ask yourself the question:

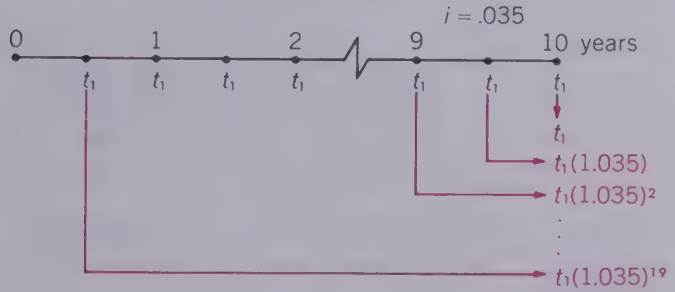
Is the lump sum of money taken (paid) now or is it taken (paid) at the end of the investment period?

Note! All annuities used in this section are ordinary annuities.

Example 1

A manufacturer sets aside a certain sum of money every six months and invests it at 7% per annum compounded semiannually. He does this for 10 years after which time he uses the money to replace a worn-out machine by purchasing a new one for $42,000. What sum did he set aside every six months to make up the replacement value?

Solution



$A = \$42{,}000$, $r = 1.035$, $n = 20$,
t_1 = money set aside every six months

$$\$42{,}000 = t_1 + t_1(1.035)^1 + t_1(1.035)^2 + \ldots + t_1(1.035)^{19}$$

Hence, for the formula

$$A = S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$\$42{,}000 = \frac{t_1(1.035^{20} - 1)}{1.035 - 1}$$

$$42{,}000 = \frac{t_1(1.98979 - 1)}{0.035}$$

$$42{,}000 \times \frac{0.035}{0.98979} = t_1$$

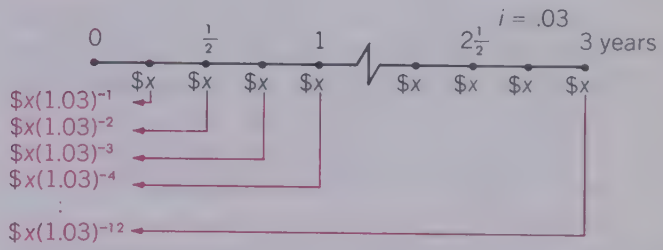
$$t_1 = \$1485.16$$

Therefore the sum set aside periodically to accrue the amount of $42,000 is $1485.16.

Example 2

Betty and Rob wished to borrow $2500 to purchase new furniture for their apartment. At the bank they were informed that the rate charged is 12% per annum compounded quarterly on payments made every three months for three years. What periodic payment would the bank require of them?

Solution



$$\$2500 = x(1.03)^{-12} + x(1.03)^{-11} + x(1.03)^{-10} + \ldots + x(1.03)^{-1}$$

$A = S_{12} = 2500$, $t_1 = x(1.03)^{-12}$, $n = 12$, $r = 1.03$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$2500 = \frac{x(1.03)^{-12}[1.03^{12} - 1]}{1.03 - 1}$$

$$2500 = \frac{x[1 - (1.03)^{-12}]}{.03}$$

$$75 = x(1 - 0.70138)$$

$$75 = 0.29862\,x$$

$$x = 251.15$$

Betty and Rob would make a payment of $251.15 every three months.

In this case $251.15 is the *periodic payment* on the *present value* of $2500.

EXERCISE 4·7

For each of the questions in this exercise:

(i) draw the time diagram,

(ii) write the sum of the series,

(iii) state the values of A, t_1, r, n,

(iv) work out the solution.

1. A contractor will require \$16,000 in three years to replace a tractor digging machine. How much money must he invest at the end of each year into an annuity which yields 8% per annum compounded annually?

2. Jane's father estimates that when Jane is 18 years old he should have saved \$4000 for her college. Jane just reached her 13th birthday. How much money will her father have to invest at the end of each year, beginning a year from now at 6% per annum compounded annually to save that amount.

3. Mr. Shurgrass requires \$8000 at present to modernize his dairy farm. He estimates that he can repay such a loan in annual payments, beginning a year from now, after four years. If he borrows the money at 7% per annum compounded annually, what will be his annual payments? What is an *ordinary* annuity?

4. What periodic payment at the end of each year is required for the present value of \$15,000 at 6% per annum compounded annually for 5 years?

5. What periodic payment is required for each of the following amounts:

   (a) \$4000 at 8% per annum compounded semiannually if payments are made every six months for 4 years?

   (b) \$12,000 at 6% per annum compounded quarterly if payments are made every three months for 5 years?

6. What periodic payment is required to repay each of the following present values?

   (a) \$3000 at 7% per annum compounded semiannually if payments are made every six months for 3 years.

   (b) \$7000 at 8% per annum compounded quarterly if payments are made every three months for 4 years.

7. Smith and Wilson want to accumulate \$25,000 in 9 years. If money can earn interest at the rate of 5% per annum compounded semiannually, how much must be invested at the end of each six-month period to accumulate this amount?

4·8 Annuities: Ordinary, Due, Deferred

All examples of annuities dealt with in the preceding sections were such that payments are made at the *end* of each annuity period (e.g., at the end of every six months). Such annuities are called *ordinary annuities.*

When payments are to be made at the beginning of each annuity interval (period) such annuities are called *annuities due*

An annuity that does not commence until a later date is called a *deferred annuity.* A man aged 40 arranges with an insurance company a payment in lump sum now for a yearly pension to begin at age 65. The annuity is then deferred for 25 years.

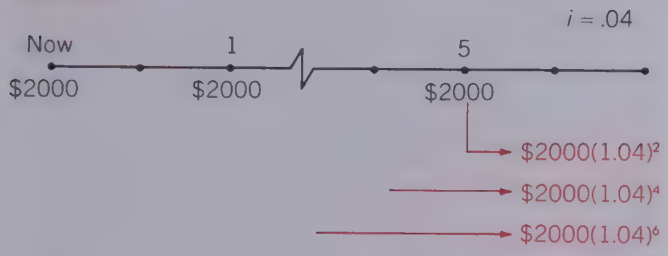
4·9 Annuities Due

At this point we will deal with an annuity where payment is made at the beginning of each time interval.

Example 1

Mr. Farmer decides to set up a reserve fund for a future need or emergency. He enrols in a plan that pays 8% interest compounded semiannually. If he deposits $2000 now, and each year thereafter, how much will the value of the fund be after the sixth deposit?

Solution



Amount = $2000\,(1.04)^2 + 2000\,(1.04)^4 + 2000\,(1.04)^6$ + . . . to 6 terms.

$t_1 = 200\,(1.04)^2$, $r = 1.04^2$, $n = 6$

$$A = \frac{t_1\,(r^n - 1)}{r - 1}$$

$$= \frac{2000\,(1.04^2)\,[(1.04^2)^6 - 1]}{1.04^2 - 1}$$

$$= \frac{2000\,(1.0816)\,[1.60103 - 1]}{1.0816 - 1}$$

$$= \frac{2163\,(0.60103)}{0.0816} = \$15{,}932 \text{ to nearest dollar}$$

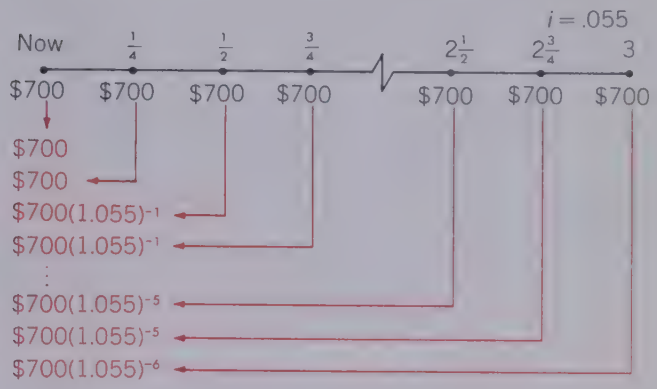
The value of the fund will be $15,932.

Example 2

Ellen requires an additional sum of money to keep open a small shop. She estimates her revenues and expects she can pay $700 every three months. She proposes to her banker a payment of $700 now and every three months thereafter for 13 such payments. On these conditions and at 11% per annum compounded semiannually how much can the banker lend her?

Solution

Here we must find the present value of this annuity.



$$P.V. = 700\,(1.055)^{-6} + 1400\,(1.055)^{-5} + 1400\,(1.055)^{-4} + \ldots + 1400\,(1.055)^{1} + 1400$$

$$= 700\,(1.055)^{-6} + \frac{1400\,(1.055)^{-5}\,[1.055^5 - 1]}{1.055 - 1} + 1400$$

$$= 700\,(0.72525) + \frac{1400\,[1 - (1.055)^{-5}]}{.055} + 1400$$

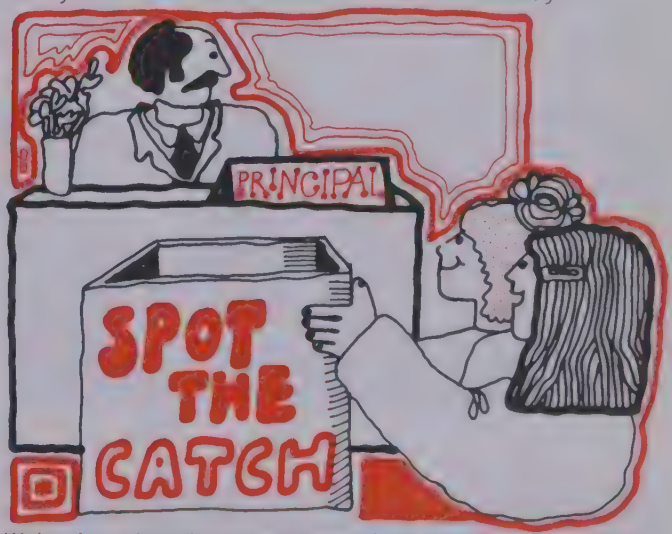
$$= 507.68 + \frac{1400\,(1 - 0.76513)}{.055} + 1400$$

$$= 507.68 + 5978.51 + 1400$$

$$= 7886.19$$

The banker can lend her $7,886.19.

Can you find the sum of this series another way?



We've just thought up and manufactured this game about the dangers of get-rich-fast schemes. Sir, now if you will just buy for $5 this introductory offer...

EXERCISE 4·9

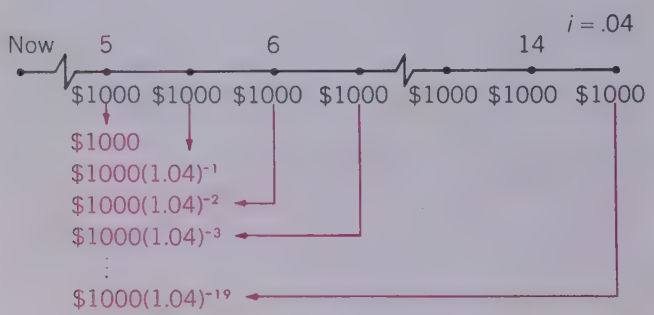
1. Payments of $1000 are made now and $1200 each year thereafter until 12 such payments are completed. If interest of 10% per annum compounded annually is levied, find
   (a) the amount of this annuity
   (b) the present value of this annuity.
2. Mr. Gregory is repaying a loan on the following terms: $800 now and $800 every six months for 20 full payments at 12% per annum compounded semiannually.
   (a) How much did Mr. Gregory borrow?
   (b) What is the amount of the annuity paid by Mr. Gregory?
3. An annuity of $500 now and $500 every three months is paid for 33 full payments. At 9% per annum compounded semiannually
   (a) what is the present value of the annuity?
   (b) what is the amount of this annuity?

4·10 Annuities Deferred

Example 1

An annuity of $1000 every six months begins five years hence and continues for 20 full payments. What is the present value of this annuity at 8% per annum compounded semiannually?

Solution



First find the value of the annuity at year 5.

$S_{20} = 1000\,(1.04)^{-19} + 1000\,(1.04)^{-18} + (1.04)^{-17} + \ldots$ to 20 terms

Note the following adjustment. Why?

$$S_{20} = [1000\,(1.04)^{-19} + 1000\,(1.04)^{-18} + \ldots + 1000\,(1.04)^{-1}] + 1000$$

$$= \frac{1000\,(1.04)^{-19}\,[1.04^{19} - 1]}{1.04 - 1} + 1000$$

$$= \frac{1000\,[1 - (1.04)^{-19}]}{.04} + 1000$$

$$= 25{,}000\,(1 - 0.47464) + 1000$$

$$= 25{,}000\,(0.52536) + 1000$$

$$= 13{,}134 + 1000$$

$$= 14{,}134$$

Then find present value of $14,134 for 5 years.

$$P.V. = \$14{,}134 \times (1.04)^{-10}$$

$$= 14{,}134 \times 0.67556$$

$$= \$9{,}548 \text{ } \textit{to nearest dollar}$$

EXERCISE 4·10

1. Mr. Black wishes to invest a lump sum of money now so that in 12 years from now he can draw his first pension cheque. He wants a pension of $2,000 every six months for 15 full years. If money is worth 7% per annum compounded semiannually, what sum of money must he invest now to earn his pension?
2. When Betty reached 10 years of age, her father invested a sum of money to provide her with an annuity for her university education. She will receive $1000 every three months for five full years beginning on her 18th birthday. The money was invested at 6% per annum compounded semiannually. What sum of money was invested for Betty?

Using the Library

1. Research the approximate time each of the following lived or happened. Record each on your time line.

   With each person's name is one or more topics. Report on the significance or interest of each topic.

   Dress up your time line with illustrations. This period is often referred to as *The Golden Age of Greek Mathematics*.

   Pythagoras – The Pythagorean Brotherhood
   Buddha
   Zeno – 4 paradoxes
   Hippocrates, Antiphon } squaring the circle
   Pericles of Athens – The Magnificent
   Hippias – Trisect an angle using ruler and compasses alone.
   Socrates – Philosopher and mathematician
   Plato – "Let no one unversed in geometry enter here."
   Aristotle – taught school.
   Alexander (The Great) – The Gordian Knot.
   Euclid – The Elements
   Eratosthenes – diameter of earth; Sieve of
   Archimedes – Spiral; "And I shall move the Earth"
   First Punic Wars
   Building of the Parthenon

2. Research the world happenings during the life of Pythagoras. What effect would these stirring times have on Pythagoras? *Hint!* Discuss cultural and literary events. Buddha, Confucius, Lao-Tze, papyrus introduced to Greece, etc.

3. Research the Pythagorean Secret Society. There was a scandal in the Society. It is reported that one person, Hippasus, "perished at sea for his impiety." What was the scandal?

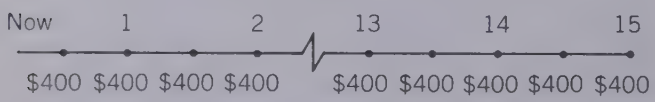
4. Research some of the anecdotes told about Thales (such as how he cured mules of submerging their loads of salt in the river). Fictionalize and write a short story of one. Illustrate pictorially another.

5. Briefly give reasons why Pythagoras left Greece and established his school at Crotona. Why was the Society dispersed? When did the dispersed Pythagoreans return to Greece?

REVIEW EXERCISE 4

1. (a) Find the simple interest on $2000 at 7% per annum for 8 years.
   (b) What is the interest on $2000 at 7% per annum compounded semiannually for 8 years?
   (c) Why is there a difference in the answers to (a) and (b)?
2. Determine the present value of $600 at 6% per annum compounded annually for 3 years.
3. Find the amount of the annuity illustrated in this time diagram.

8% per annum compounded quarterly



4. Find the present value of the annuity illustrated by the time diagram in question 3.
5. James deposits $80 every three months beginning now for 21 such deposits at 7% per annum compounded semiannually. How much is in James' account after the 21st deposit?

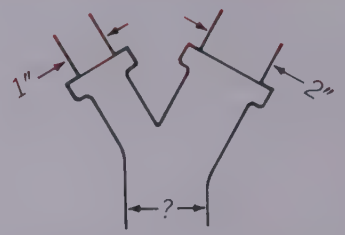
Your Job and Mathematics

The Modern Building Co. Ltd.

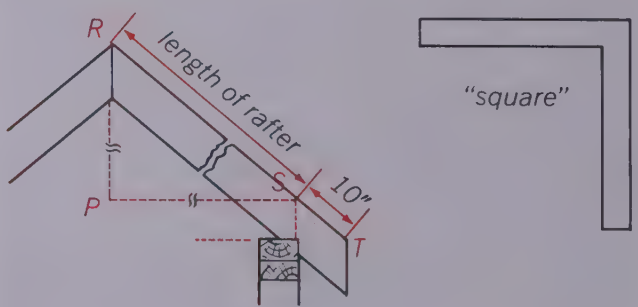
White Rock, B.C.

A company involved with building trades often has subdivisions within the company to handle specific building problems. The three problems illustrated here involve carpentry, plumbing and masonry. Each tradesman, whether he be a carpenter, mason or plumber spends years working as an apprentice acquiring the necessary skills for the trade. Once fully qualified he draws good wages and is in demand. A plumber may earn $8.00 per hour, a mason and a carpenter $6.50.

1. A plumber in building a water line has a 2" pipe (2-inch diameter) and a 1" pipe joining into a large pipe. He must calculate the minimum size of the third pipe so that it will carry the full capacity of the two smaller ones.
   Hint! The carrying capacity of a pipe varies directly with the cross section area of the pipe.



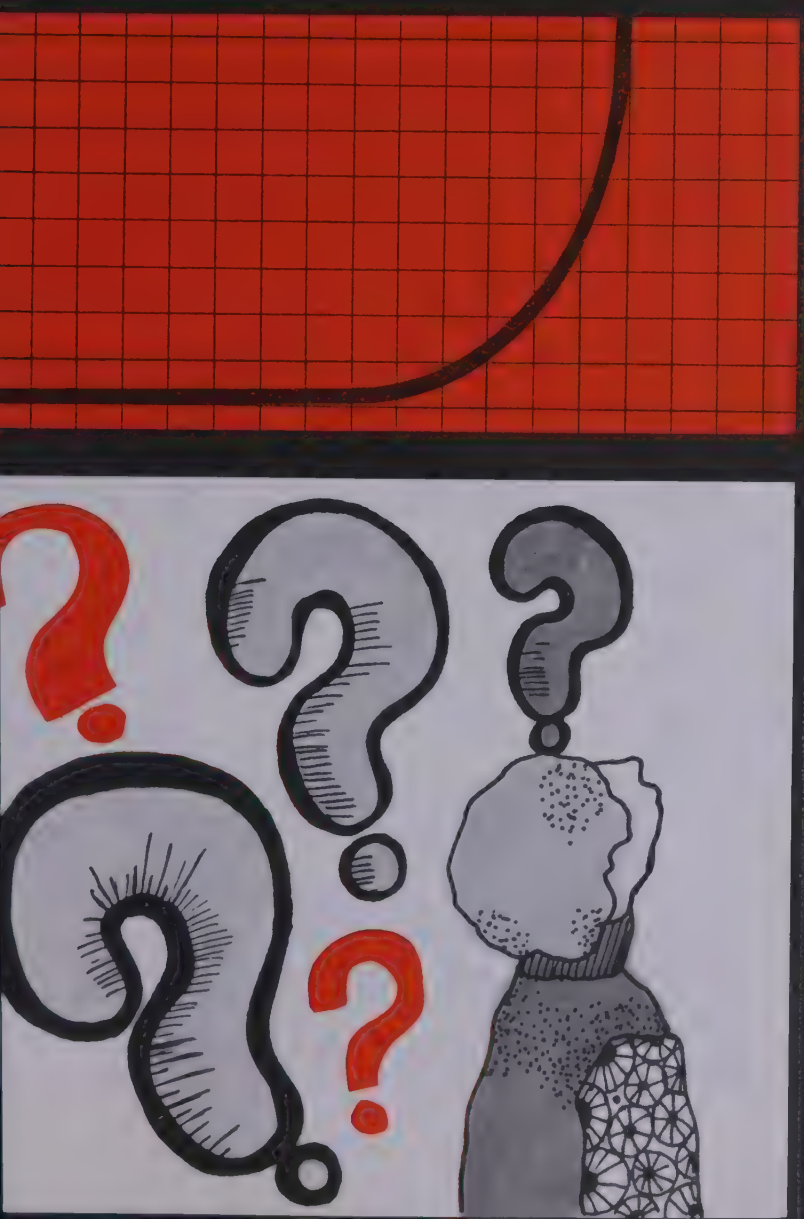
2. Bricks, measuring 4" × 8" including the mortar (the cement between the bricks) are used to face a building 64′ wide and 22′ high. There is a door 4′ × 7′ and two windows 5′ × 8′. Three types of bricks are to be used in a definite pattern. There are to be twice as many white bricks as dull red, and three times as many drip style light orange-red as dull red. Calculate the number of each type needed, allowing 2% for breakage and edges.
3. One of the most important jobs, yet difficult to the beginner carpenter, is cutting of rafters to the correct length. After he has learned to solve this type of problem shown below so that he knows the theory, he is shown how to use a *square* to cut the rafters. Can you find the *length of the rafters* and the overall length of the rafter allowing for the eaves?

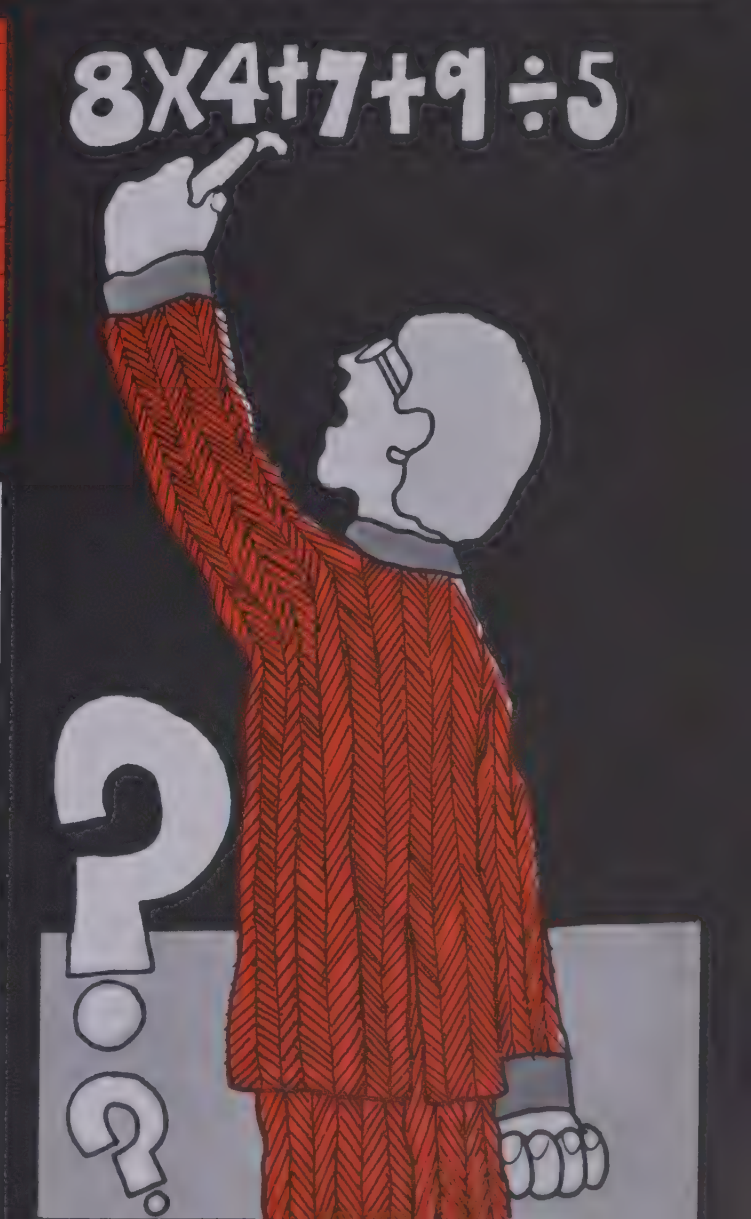


   The run is 10 feet ($\overline{SP}$). The rise is $3\frac{1}{2}$ inches per foot of run. Calculate the angle of the cut, $\angle SRP$.
4. Invite a carpenter to explain how to use a square to cut rafters.

Chapter Five

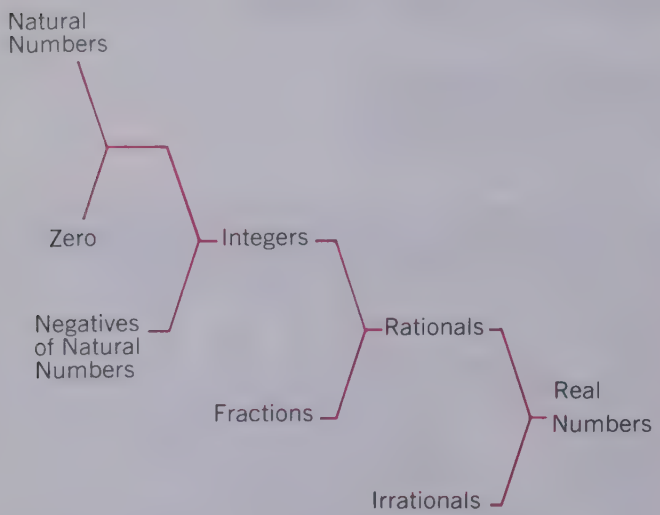
RADICALS, EXPONENTS, AND LOGARITHMS





5·1 Irrationals

Throughout human history man invented and developed number sets as the needs of the time dictated. The following illustration briefly sketches the progress in the development of the real number set.



But exactly what is the set of irrational numbers? Let us consider the following example.

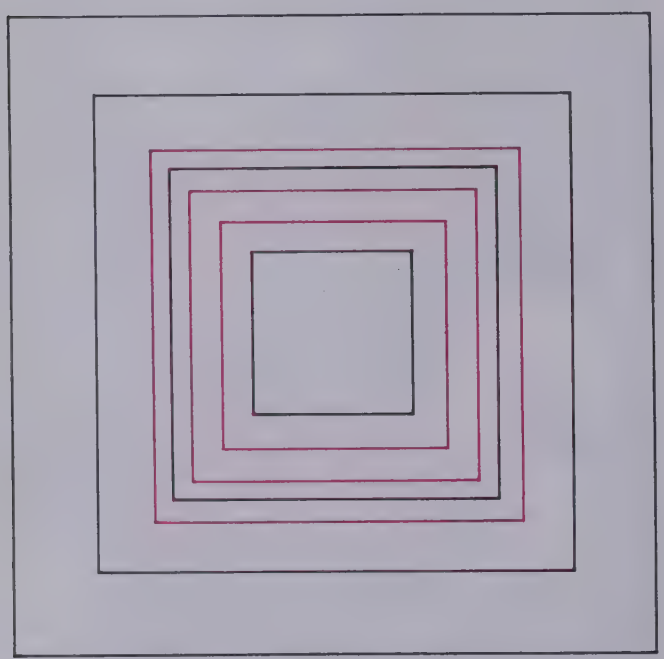


Figure 5-1

Figure 5-1 is a diagram of squares. The innermost square is 1 square inch in area. The next larger one is 2 square inches in area. The third is 3, the fourth 4, the fifth 5, the sixth 9, and the largest 16 square inches in area. For the squares 1, 4, 9 and 16 square inches in area it is readily deduced that the sides of the corresponding squares are 1, 2, 3 and 4 inches long. For the squares which are 2, 3 and 5 square inches in area, it is not simple to express the lengths of the corresponding sides. After some reasoning we may decide to express these lengths as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ inches for the 2, 3 and 5 square-inch squares respectively.

Numbers like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$, which cannot be expressed in fraction form, in repeating decimals nor in terminating decimals are called *irrational numbers*.

From the squares in Figure 5-1, as well as from Figure 5-2, we observe that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ etc. are measured units of length. Therefore, it is correct to conclude that there are corresponding points on the number line for irrational numbers. $\sqrt{7}$ would correspond to some point between 2 and 3 on the number line.

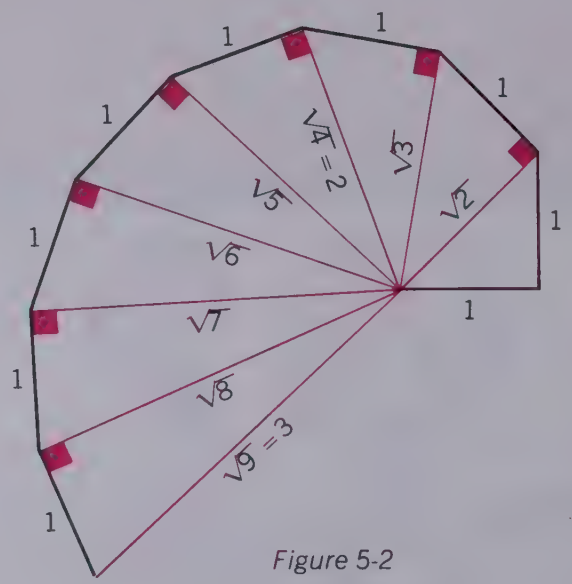


Figure 5-2

Irrationals written in the form as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. (and not as nonterminating decimals) are called *radicals*. Irrationals can never be expressed in equivalent decimal form, only in nonterminating and nonrepeating decimal form. Rationals may also be written as radicals (e.g., $\sqrt{16}$, $\sqrt[3]{27}$, $\sqrt[5]{32}$).

EXERCISE 5·1

1. Which of the following radicals are rationals and which are irrationals?

$\sqrt{25}$, $\sqrt[3]{8}$, $\sqrt{80}$, $\sqrt{125}$, $\sqrt[3]{12}$, $\sqrt[4]{81}$, $\sqrt[3]{-27}$, $\sqrt[5]{125}$, $\sqrt[4]{16}$, $\sqrt[3]{-125}$, $\sqrt{144}$, $\sqrt[5]{243}$, $\sqrt[3]{36}$, $\sqrt{50}$, $\sqrt[4]{625}$

5·2 Operations With Radicals (Review)

The addition property:

(i) $\sqrt{a} + \sqrt{a} = 2\sqrt{a}$

(ii) $a\sqrt{x} + b\sqrt{x} = (a + b)\sqrt{x}$

The subtraction property:

(i) $\sqrt{m} - \sqrt{m} = 0$

(ii) $r\sqrt{t} - s\sqrt{t} = (r - s)\sqrt{t}$

The multiplication property:

(i) $\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}$

(ii) $a\sqrt{b} \cdot m\sqrt{n} = am\sqrt{bn}$

The division property:

(i) $\dfrac{\sqrt{m}}{\sqrt{n}} = \sqrt{\dfrac{m}{n}}$ (ii) $\dfrac{a\sqrt{c}}{r\sqrt{s}} = \dfrac{a}{r}\sqrt{\dfrac{c}{s}}$

Example 1

Simplify: (i) $7\sqrt{3} + 5\sqrt{3} - 8\sqrt{3}$,

(ii) $2\sqrt{5} - 3\sqrt{2} + 5\sqrt{5} + \sqrt{2} - 3\sqrt{5}$.

Solution

(i) $7\sqrt{3} + 5\sqrt{3} - 8\sqrt{3}$
$= (7 + 5 - 8)\sqrt{3}$ *distributive*
$= 4\sqrt{3}$

(ii) $2\sqrt{5} - 3\sqrt{2} + 5\sqrt{5} + \sqrt{2} - 3\sqrt{5}$
$= 2\sqrt{5} + 5\sqrt{5} - 3\sqrt{5} - 3\sqrt{2} + \sqrt{2}$
associative
$= (2 + 5 - 3)\sqrt{5} + (-3 + 1)\sqrt{2}$
distributive
$= 4\sqrt{5} - 2\sqrt{2}$

Without mathematics involving irrationals, man could not have progressed to his present state of civilization.

Example 2

Evaluate: (i) $\sqrt{120} \times \sqrt{30}$,
(ii) $5\sqrt{8} \times 3\sqrt{3} \times \sqrt{6}$.

Solution

(i) $\sqrt{120} \times \sqrt{30} = \sqrt{3600}$
$= 60$

or

$\sqrt{120} \times \sqrt{30} = \sqrt{4 \cdot 30} \times \sqrt{30}$
$= 2\sqrt{30} \times \sqrt{30}$
$= 2 \times 30$
$= 60$

(ii) $5\sqrt{8} \times 3\sqrt{3} \times \sqrt{6} = 15\sqrt{24} \times \sqrt{6}$
$= 15\sqrt{4 \cdot 6} \times \sqrt{6}$
$= 180$

or

$15\sqrt{144} = 15 \times 12$
$= 180$

Example 3

(i) Evaluate $\sqrt{180} \div \sqrt{20}$.

(ii) Simplify $\sqrt{108} \div \sqrt{180}$.

Solution

(i) $\dfrac{\sqrt{180}}{\sqrt{20}} = \sqrt{\dfrac{180}{20}} = \sqrt{9} = 3$

(ii) $\dfrac{\sqrt{108}}{\sqrt{180}} = \sqrt{\dfrac{108}{180}} = \sqrt{\dfrac{36 \times 3}{36 \times 5}} = \sqrt{\dfrac{3}{5}}$

EXERCISE 5·2

1. Simplify:
   (a) $2\sqrt{3} - 7\sqrt{3} + 5\sqrt{3}$ (b) $\sqrt{20} + 3\sqrt{5}$
   (c) $5\sqrt{8} - 7\sqrt{2}$ (d) $4\sqrt{63} + 3\sqrt{28}$
   (e) $4\sqrt{6} + 2\sqrt{5} + 7\sqrt{6} - 8\sqrt{5}$
   (f) $2\sqrt{32} - \sqrt{75} - \sqrt{8} + 2\sqrt{12}$
   (g) $5\sqrt{28} - 2\sqrt{20} - 4\sqrt{45} - 2\sqrt{63}$.

2. Evaluate:
   (a) $\sqrt{12} \times \sqrt{27}$ (b) $\sqrt{84} \times \sqrt{21}$
   (c) $\sqrt{60} \times \sqrt{15}$ (d) $\sqrt{35} \times \sqrt{10} \times \sqrt{14}$
   (e) $\sqrt{126} \times \sqrt{56}$ (f) $\sqrt{39} \times \sqrt{15} \times \sqrt{65}$
   (g) $\sqrt{56} \times \sqrt{154} \times \sqrt{176}$ (h) $\sqrt{27} \div \sqrt{3}$
   (i) $\sqrt{80} \div \sqrt{5}$ (j) $\sqrt{128} \div \sqrt{8}$

3. Simplify:
   (a) $3\sqrt{7} \times 2\sqrt{3}$ (b) $2\sqrt{5} \times 3\sqrt{15} \times 4\sqrt{20}$
   (c) $8\sqrt{8} \times 5\sqrt{3}$ (d) $10\sqrt{12} \times 9\sqrt{18} \times 2\sqrt{30}$
   (e) $\sqrt{52} \div \sqrt{8}$ (f) $4\sqrt{70} \div \sqrt{14}$
   (g) $15\sqrt{144} \div 3\sqrt{48}$ (h) $24\sqrt{400} \div 6\sqrt{80}$

5·3 Rationalizing the Denominator

To rationalize the denominator means to write the expression in an equivalent form but with the denominator as a rational number.

Example 1

Rationalize the denominator and simplify:

(a) $\frac{3}{\sqrt{2}}$ (b) $\frac{5\sqrt{2}}{\sqrt{3}}$ (c) $\frac{2\sqrt{7}}{3\sqrt{8}}$ (d) $\frac{5\sqrt{6}}{8\sqrt{24}}$.

Solution

(a) $\frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ (b) $\frac{5\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{6}}{3}$

(c) $\frac{2\sqrt{7}}{3\sqrt{8}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{14}}{3\sqrt{16}} = \frac{2\sqrt{14}}{12} = \frac{\sqrt{14}}{6}$

(d) $\frac{5\sqrt{6}}{8\sqrt{24}} = \frac{5}{8\sqrt{4}} = \frac{5}{16}$

or $\frac{5\sqrt{6}}{8\sqrt{24}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{30}{8\sqrt{144}} = \frac{30}{8 \times 12} = \frac{5}{16}$

Do question 1 in Exercise 5·3.

Notice that in the question all the denominators are irrationals: $\sqrt{2}, \sqrt{3}, 3\sqrt{8}, 8\sqrt{24}$ but in the answers the denominators are all rational numbers: 2, 3, 6, 16. We have rationalized the denominators.

Sometimes the denominator of an irrational expression is the sum or difference of two radicals, for example $\frac{1}{\sqrt{3} - \sqrt{2}}$ or $\frac{1}{\sqrt{3} + \sqrt{5}}$.

In order to rationalize such denominators we need to learn about the *conjugate* of a radical expression.

Do question 2 in Exercise 5·3.

Example 2

Rationalize the denominator and simplify:

(a) $\frac{2 - \sqrt{5}}{2 + \sqrt{3}}$ (b) $\frac{5 - \sqrt{6}}{\sqrt{2} - \sqrt{3}}$ (c) $\frac{2\sqrt{3} + 5\sqrt{5}}{3\sqrt{5} - 4\sqrt{3}}$.

Solution

(a) The conjugate expression for the denominator is $2 - \sqrt{3}$. Multiply fraction by $\frac{2 - \sqrt{3}}{2 - \sqrt{3}}$.
Thus

$$\frac{2 - \sqrt{5}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{4 - 2\sqrt{3} - 2\sqrt{5} + \sqrt{15}}{4 - 2\sqrt{3} + 2\sqrt{3} - 3}$$
$$= \frac{4 - 2\sqrt{3} - 2\sqrt{5} + \sqrt{15}}{1}$$
$$= 4 - 2\sqrt{3} - 2\sqrt{5} + \sqrt{15}$$

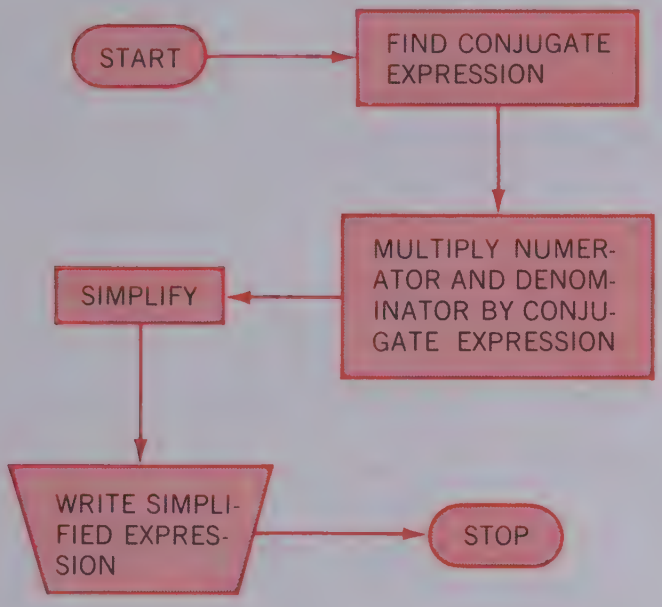
(b) The conjugate expression for the denominator is $\sqrt{2} + \sqrt{3}$.
Thus $\frac{5 - \sqrt{6}}{\sqrt{2} - \sqrt{3}} = \frac{5 - \sqrt{6}}{\sqrt{2} - \sqrt{3}} \times \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} + \sqrt{3}}$

$$= \frac{5\sqrt{2} + 5\sqrt{3} - \sqrt{12} - \sqrt{18}}{2 - 3}$$
$$= \frac{5\sqrt{2} + 5\sqrt{3} - 2\sqrt{3} - 3\sqrt{2}}{-1}$$
$$= \frac{2\sqrt{2} + 3\sqrt{3}}{-1} = -2\sqrt{2} - 3\sqrt{3}$$

(c) $\dfrac{2\sqrt{3}+5\sqrt{5}}{3\sqrt{5}-4\sqrt{3}} = \dfrac{2\sqrt{3}+5\sqrt{5}}{3\sqrt{5}-4\sqrt{3}} \times \dfrac{3\sqrt{5}+4\sqrt{3}}{3\sqrt{5}+4\sqrt{3}}$

$= \dfrac{6\sqrt{15}+24+75+20\sqrt{15}}{45-48}$

$= \dfrac{99+26\sqrt{15}}{-3} = -\dfrac{99+26\sqrt{15}}{3}$



EXERCISE 5·3

1. Rationalize the denominator and simplify:

   (a) $\dfrac{2}{\sqrt{5}}$ (b) $\dfrac{6}{\sqrt{6}}$ (c) $\dfrac{14}{\sqrt{7}}$ (d) $\dfrac{8}{\sqrt{12}}$

   (e) $\dfrac{\sqrt{3}}{\sqrt{18}}$ (f) $\dfrac{\sqrt{5}}{2\sqrt{3}}$ (g) $\dfrac{4\sqrt{3}}{5\sqrt{8}}$ (h) $\dfrac{25\sqrt{2}}{\sqrt{50}}$.

2. Find the product of each:

   (a) $(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})$ (b) $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$

   (c) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{5})$ (d) $(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})$

   (e) $(\sqrt{5}+\sqrt{7})(\sqrt{5}+\sqrt{7})$ (f) $(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})$

   (g) $(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{b})$ (h) $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$

   (i) What do you notice about the products of (b), (d), (f), (h) compared with the products of (a), (c), (e), (g)?

   (j) How are parts (b), (d), (f) and (h) related to the product $(a+b)(a-b)$?

   (k) Two radical expressions whose product is a rational number are called *conjugate radicals*. Hence $(3\sqrt{5}-\sqrt{11})$ and $(3\sqrt{5}+\sqrt{11})$ are conjugate radicals. Explain.

   (l) Write the conjugate radical expression for each:
   $\sqrt{7}+\sqrt{8}$, $2\sqrt{5}-\sqrt{8}$, $\sqrt{3}-2$, $4-\sqrt{5}$, $a-\sqrt{b}$, $\sqrt{a}+5$, $3\sqrt{x}-2\sqrt{y}$, $a\sqrt{w}+\sqrt{m}$.

3. Rationalize the denominator and simplify:

   (a) $\dfrac{5}{2-\sqrt{3}}$ (b) $\dfrac{8}{\sqrt{5}+\sqrt{2}}$

   (c) $\dfrac{4\sqrt{3}}{\sqrt{8}-2\sqrt{3}}$ (d) $\dfrac{3\sqrt{5}+2\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

   (e) $\dfrac{5\sqrt{8}-2\sqrt{12}}{3\sqrt{2}+2\sqrt{3}}$ (f) $\dfrac{8\sqrt{7}+2\sqrt{5}}{6\sqrt{5}-3\sqrt{7}}$

5·4 Simple Radical Equations

An equation with a variable under a radical sign as one of the terms is a *radical equation*. An example of a radical equation is $\sqrt{5x+1}-4=0$.

To solve a radical equation one must isolate the term with the radical sign and then square (or cube as the case may be) each side of the equation. Observe the procedures in the examples that follow.

Example 1

Solve for n: $n \in N$.

(a) $\sqrt{3n}=9$ (b) $\sqrt{4n}=8$

Solution

(a) $\sqrt{3n}=9$

Square both sides of equation:

$3n=81$

$n=\frac{81}{3}=27$

(b) $\sqrt{4n}=8$

Square both sides of equation:

$4n=64$

$n=\frac{64}{4}=16$

Example 2

Solve for z:

(a) $\sqrt{z+1} = 2$ (b) $\sqrt{7-2z} = 3$

Solution

(a)
$$\sqrt{z+1} = 2$$
$$(\sqrt{z+1})^2 = 2^2 \quad \textit{(Remember! } (\sqrt{a})^2 = a \textit{ and } (\sqrt{a+1})^2 = a+1\textit{)}$$
$$z+1 = 4$$
$$z = 3$$

(b)
$$\sqrt{7-2z} = 3$$
$$(\sqrt{7-2z})^2 = 3^2$$
$$7-2z = 9$$
$$-2z = 2$$
$$z = -1$$

In solving radical equations it is always necessary to verify the answers. Note the next example. Can you explain why one of the answers will not verify?

Example 3

Solve for x: $\sqrt{3x+1} = x-1$

Solution

$$\sqrt{3x+1} = x-1$$
$$(\sqrt{3x+1})^2 = (x-1)^2 \quad \textit{Why?}$$
$$3x+1 = x^2-2x+1 \quad \textit{Did you follow this?}$$
$$0 = x^2-5x$$
$$0 = x(x-5)$$
$$x = 0 \text{ or } x-5 = 0 \quad \textit{Why?}$$
$$x = 5$$

Verification

For $x = 0$	For $x = 5$
$L.H.S. = \sqrt{3(0)+1}$	$L.H.S. = \sqrt{3(5)+1}$
$= \sqrt{1}$	$= \sqrt{16}$
$= 1$	$= 4$
$R.H.S. = 0-1$	$R.H.S. = 5-1$
$= -1$	$= 4$
$\therefore x = 0$ is not in the solution set.	$\therefore x = 5$ is in the solution set.

The solution set is $\{5\}$.

Example 4

Solve for x: $1 - \sqrt{x} = \sqrt{2x+1}$, $x \geq 0$.

Solution

$$1 - \sqrt{x} = \sqrt{2x+1}$$
$$(1-\sqrt{x})^2 = (\sqrt{2x+1})^2 \quad \textit{What does this operation achieve?}$$
$$1 - 2\sqrt{x} + x = 2x+1$$
$$-2\sqrt{x} = x$$
$$(-2\sqrt{x})^2 = x^2 \quad \textit{Why?}$$
$$4x = x^2$$
$$0 = x^2 - 4x$$
$$x(x-4) = 0$$
$$x = 0 \text{ or } x-4 = 0$$
$$x = 4$$

Verification

For $x = 0$	For $x = 4$
$L.H.S. = 1 - \sqrt{0}$	$L.H.S. = 1 - \sqrt{4}$
$= 1 - 0$	$= 1 - 2$
$= 1$	$= -1$
$R.H.S. = \sqrt{2(0)+1}$	$R.H.S. = \sqrt{2(4)+1}$
$= \sqrt{1}$	$= \sqrt{9}$
$= 1$	$= 3$
	Does not verify. Therefore $x = 4$ is not a solution.

$\therefore$ Solution set is $\{0\}$.

EXERCISE 5·4

1. Solve each of the following. Verify each.
   (a) $\sqrt{n} = 7$ (b) $\sqrt{25x} = 15$
   (c) $\sqrt{x-1} = 3$ (d) $\sqrt{2z+4} = 4$
   (e) $\sqrt{7+3n} = 4$ (f) $\sqrt{31-2z} = 5$
   (g) $\sqrt[3]{x} = 2$ (h) $\sqrt[3]{5x+14} = 4$

2. Solve for the variable in each case. Verify each.
   (a) $\sqrt{6x-8} = x$ (b) $x = \sqrt{3+2x}$
   (c) $n = \sqrt{5n+14}$ (d) $2n = \sqrt{11n+3}$

3. Find the solution set of each equation. Verify each.
   (a) $\sqrt{3x-9} = \sqrt{5+x}$
   (b) $\sqrt{2z+1} = \sqrt{z}+1$
   (c) $\sqrt{k-5} = \sqrt{k}-1$
   (d) $\sqrt{4s} = \sqrt{s}+3$
   (e) $\sqrt{x-8} = 2-\sqrt{x}$
   (f) $\sqrt{4p-7} - \sqrt{2p} = 1$
   (g) $\sqrt{13n-1} = 5+\sqrt{3n-6}$
   (h) $\sqrt{4x+5} = 9-\sqrt{x-7}$
   (i) $\sqrt{2x-2} - \sqrt{x-3} = 2$
   (j) $\sqrt{8n+1} - 3 = 2\sqrt{n-1}$

5·5 Power Laws

The expression b^a is called a power, where b is called the base and a is called the exponent.

Example 1

(a) Write 25 as a power of 5.
(b) Write 16 as a power of 2.

Solution

(a) $25 = 5^2$
(b) $16 = 2^4$

Multiplication Rule for Powers

$$b^a \times b^c = b^{a+c}$$

Division Rule for Powers

$$b^a \div b^c = b^{a-c}$$

Example 2

Simplify:
(a) $3^5 \times 3^2$ (b) $7^4 \times 7^8$
(c) $x^2 \times x^3 \times x^5$ (d) $2m^4 \times 6m^5$
(e) $8^5 \div 8^2$ (f) $24r^7 \div 8r^3$

Solution

(a) $3^5 \times 3^2 = 3^{5+2} = 3^7$
(b) $7^4 \times 7^8 = 7^{4+8} = 7^{12}$
(c) $x^2 \times x^3 \times x^5 = x^{2+3+5} = x^{10}$
(d) $2m^4 \times 6m^5 = 12m^9$
(e) $8^5 \div 8^2 = 8^{5-2} = 8^3$
(f) $24r^7 \div 8r^3 = \frac{24}{8}r^{7-3} = 3r^4$

Do questions 1 through 8 in Exercise 5·5.

Power Rule for Exponents

$$(b^a)^c = b^{ac}$$

Power of a Product

$$(bh)^a = b^a h^a$$

Power of a Quotient

$$\left(\frac{b}{h}\right)^c = \frac{b^c}{h^c}$$

Example 3

Simplify:

(a) $(5^3)^2$ (b) $(x^8)^5$

(c) $(3m)^3$ (d) $(2a^2b)^5$

(e) $\left(\frac{2x^3}{3y}\right)^2$

Solution

(a) $(5^3)^2 = 5^{3 \cdot 2} = 5^6$

(b) $(x^8)^5 = x^{8 \cdot 5} = x^{40}$

(c) $(3m)^3 = 3^3m^3 = 27m^3$

(d) $(2a^2b)^5 = 2^5a^{10}b^5 = 32\,a^{10}b^5$

(e) $\left(\frac{2x^3}{3y}\right)^2 = \frac{2^2x^6}{3^2y^2} = \frac{4x^6}{9y^2}$

STOP Do questions 9 through 21 in Exercise 5·5.

Find examples in your science course where exponents are used. Make a bulletin board display.

$$b^0 = 1,\ b \neq 0$$

$$b^{-a} = \frac{1}{b^a},\ b \neq 0$$

Example 4

Evaluate:

(a) $3x^0$ (b) $4y^6 \times 2y^{-5} \times y^{-1}$

(c) 5^{-3} (d) $\frac{12m^3}{4m^{-2}}$

Solution

(a) $3x^0 = 3(1) = 3$

(b) $4y^6 \times 2y^{-5} \times y^{-1} = 8y^{6+(-5)+(-1)}$
$= 8y^0 = 8(1) = 8$

(c) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(d) $\frac{12m^3}{4m^{-2}} = 3m^{3-(-2)} = 3m^5$

EXERCISE 5·5

Simplify each of the following:

1. $a^2 \times a^7 \times a^3$
2. $4^2 \times 2^3$
3. $8^5 \times 16^3$
4. $27^4 \times 81^2$
5. $3b^4 \times 7b^6$
6. $z^5 \times 4z^3 \times 6z$
7. $12x \div 3$
8. $15a^2 \div 5a$
9. $32b^7 \div (2b)^3$
10. $(5y^2)^5 \div 25y^6$
11. $(9k^3)^4 \div (3k)^5$
12. $(4x^7)^5 \div 64x^{20}$
13. $\frac{(3x^2)^3}{9x^6}$
14. $\frac{(a^2b)^3}{a^3b^2}$
15. $(-3)^4$
16. $4x^6 \div (2x^3)^2$
17. $(-1)^{17}$
18. $-36a^2b^5 \div (3a^2b)^2$
19. $(b^x)^x$
20. $-(3x^2)^0$
21. $\left(\frac{3k}{2m^2}\right)^4$
22. $\frac{32^x}{8^{2x}}$
23. $\frac{5^a \times 125^{3a}}{25^{6a}}$
24. $\left(\frac{m}{n}\right)^5 - \left(\frac{m}{n}\right)^{-2}$

5·6 Power Laws for Rational Exponents (Optional)

$b^{\frac{1}{2}} = \sqrt{b}, b \geq 0$

$b^{\frac{1}{3}} = \sqrt[3]{b}$

$b^{\frac{1}{n}} = \sqrt[n]{b}, b \geq 0$ when n is even

$b^{-\frac{1}{n}} = \dfrac{1}{\sqrt[n]{b}}, b > 0$ when n is even and $b \neq 0$.

Example 1

Evaluate:

(a) $49^{\frac{1}{2}}$ (b) $64^{\frac{1}{3}}$ (c) $32^{\frac{1}{5}}$

(d) $16^{-\frac{1}{4}}$ (e) $(-27)^{\frac{1}{3}}$

Solution

(a) $49^{\frac{1}{2}} = \sqrt{49} = 7$ (b) $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

(c) $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ (d) $16^{-\frac{1}{4}} = \dfrac{1}{\sqrt[4]{16}} = \frac{1}{2}$

(e) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

$b^{\frac{a}{c}} = \sqrt[c]{b^a} = (\sqrt[c]{b})^a$

$b^{-\frac{a}{c}} = \dfrac{1}{\sqrt[c]{b^a}} = \dfrac{1}{(\sqrt[c]{b})^a}$

Example 2

Evaluate:

(a) $8^{\frac{4}{3}}$ (b) $36^{\frac{3}{2}}$ (c) $16^{-\frac{3}{4}}$ (d) $125^{-\frac{2}{3}}$

Solution

(a) $8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$

(b) $36^{\frac{3}{2}} = (\sqrt[2]{36})^3 = 6^3 = 216$ *Note!* $\sqrt[2]{36} = 6$

(c) $16^{-\frac{3}{4}} = \dfrac{1}{(\sqrt[4]{16})^3} = \dfrac{1}{2^3} = \frac{1}{8}$ *Note!* $\sqrt[4]{16} = 2$

(d) $125^{-\frac{2}{3}} = \dfrac{1}{(\sqrt[3]{125})^2} = \dfrac{1}{5^2} = \frac{1}{25}$

Example 3

Evaluate:

(a) $4^{\frac{5}{2}} \times 32^{\frac{3}{5}}$ (b) $8^{\frac{4}{3}} \div 4^{\frac{1}{2}}$ (c) $32^{\frac{6}{5}} \times 64^{-\frac{1}{2}}$

Solution

(a) $4^{\frac{5}{2}} \times 32^{\frac{3}{5}}$

$= (2^2)^{\frac{5}{2}} \times (\sqrt[5]{32})^3$ *Note!* $(2^2)^{\frac{5}{2}} = 2^{2 \cdot \frac{5}{2}} = 2^5$

$= 2^5 \times 2^3 = 2^8 = 256$

(b) $8^{\frac{4}{3}} \div 4^{\frac{1}{2}} = \dfrac{(\sqrt[3]{8})^4}{\sqrt{4}} = \dfrac{2^4}{2} = 2^3 = 8$

(c) $32^{\frac{6}{5}} \times 64^{-\frac{1}{2}} = (\sqrt[5]{32})^6 \times \dfrac{1}{\sqrt{64}}$

$= 2^6 \times \frac{1}{8} = \dfrac{2^6}{2^3} = 2^3 = 8$

EXERCISE 5·6

1. State simplified or equivalent forms of each of the following.

(a) $x^{\frac{1}{2}}$ (b) $8^{\frac{2}{3}}$ (c) $(3^2)^{-\frac{1}{3}}$

(d) $7^{\frac{1}{4}}$ (e) $\dfrac{1}{m^{-\frac{1}{2}}}$ (f) $(a^3)^{\frac{5}{9}}$

(g) $a^{-\frac{1}{2}}$ (h) $(x^3)^{\frac{1}{2}}$

2. Evaluate:

(a) $8^{\frac{4}{3}}$ (b) $125^{\frac{2}{3}}$ (c) $(-27)^{\frac{2}{3}}$

(d) $81^{\frac{3}{4}}$ (e) $\left(\dfrac{4}{25}\right)^{\frac{1}{2}}$ (f) $(-32)^{\frac{3}{5}}$

(g) $64^{\frac{5}{6}}$ (h) $343^{-\frac{1}{3}}$ (i) $(-8)^{-\frac{2}{3}}$

(j) $27^{-\frac{2}{3}}$ (k) $128^{-\frac{2}{7}}$ (l) $\left(\dfrac{1}{400x^6}\right)^{-\frac{1}{2}}$

3. Evaluate:

(a) $3^{\frac{1}{2}} \times 3^{\frac{3}{2}} \times 3^{-1}$ (b) $\dfrac{4^{\frac{2}{3}} \times 4^{\frac{5}{6}}}{4^{\frac{5}{12}}}$

(c) $\dfrac{a^{\frac{3}{5}} \times a^{\frac{3}{10}}}{a^{\frac{4}{15}}}$ (d) $\dfrac{25^{\frac{3}{2}} \times 4(125^{\frac{2}{3}})}{125^{\frac{4}{3}}}$

(e) $\dfrac{81^{\frac{3}{4}} \times 27^{\frac{2}{3}}}{9^{\frac{5}{2}}}$ (f) $x^{\frac{2}{5}} \cdot x^{\frac{3}{10}} \cdot x^{-\frac{1}{5}}$

5·7 Logarithms

Let us take the expression $100 = 10^2$.

An equivalent expression is $\log_{10}100 = 2$.

We read it *Logarithm of 100 to base 10 is 2*

Similarly, for $81 = 3^4$ we have an equivalent expression: $\log_3 81 = 4$. And for $64 = 2^6$ we write $\log_2 64 = 6$.

In general, we may state

if $b = a^x$ (exponential form)

then $\log_a b = x$ (logarithmic form).

EXERCISE 5·7

1. Write each of the following in logarithmic form.

   (a) $8 = 2^3$ (b) $25 = 5^2$

   (c) $9^2 = 81$ (d) $16 = 4^2$

   (e) $10{,}000 = 10^4$ (f) $2^7 = 128$

   (g) $125 = 5^a$ (h) $8^n = 64$

   (i) $p = 3^m$ (j) $y = x^a$

2. Write each of the following in exponential form.

   (a) $\log_{10}100{,}000 = 5$ (b) $\log_3 243 = 5$

   (c) $\log_7 343 = 3$ (d) $\log_5 625 = 4$

   (e) $\log_3 6561 = 8$ (f) $\log_4 1024 = 5$

   (g) $\log_2 32 = x$ (h) $\log_8 64 = n$

   (i) $\log_m k = h$ (j) $\log_b m = a$

5·8 Logarithms With Base Ten

From Section 5·7 we noted that the logarithm of a number is the exponent. We also noted that the logarithm of a certain number changes when the base changes.

For example, the $\log_2 64 = 6$, while $\log_4 64 = 3$ and $\log_8 64 = 2$. To verify this write each of the three logarithmic forms in exponential form.

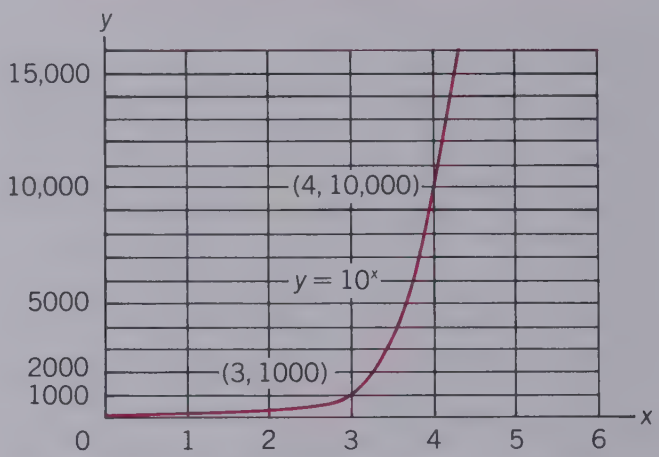
Let us examine the possibility of writing the logarithms of all positive rational numbers to the same base. In our decimal number system a logical base to choose is 10. Unfortunately, it is not easy to find exponents (with base 10) of numbers like 6, 40, 183, 2670, etc.

If we approach the problem by graphing

$$y = 10^x$$

we might realize that for each positive number y, there is a corresponding exponent x when taken to base 10. A table of values for $y = 10^x$ is shown.

x	0	1	2	3	4
y	1	10	100	1000	10,000



We immediately notice that there are many values of y for which we cannot readily find the corresponding value of x. However, when we study the graph of $y = 10^x$ (Figures 5-4 and 5-5) we realize that there are distinct values of x for each value of y. For example, if y is 20, an approximate corresponding value of x is 1.3. Other approximate values are shown in this table.

y	1	5	8	10	20	60	80	100
x	0	.7	.9	1	1.3	1.8	1.9	2

In other words we note

$20 \simeq 10^{1.3}$ $60 \simeq 10^{1.8}$ $80 \simeq 10^{1.9}$

Therefore

$\log_{10}20 \simeq 1.3$ $\log_{10}60 \simeq 1.8$ $\log_{10}80 \simeq 1.9$

The logarithm 1.8 has two sub-labels. The 1 (or whole part) is called the *characteristic* while the .8 (or fractional part) is called the *mantissa.*

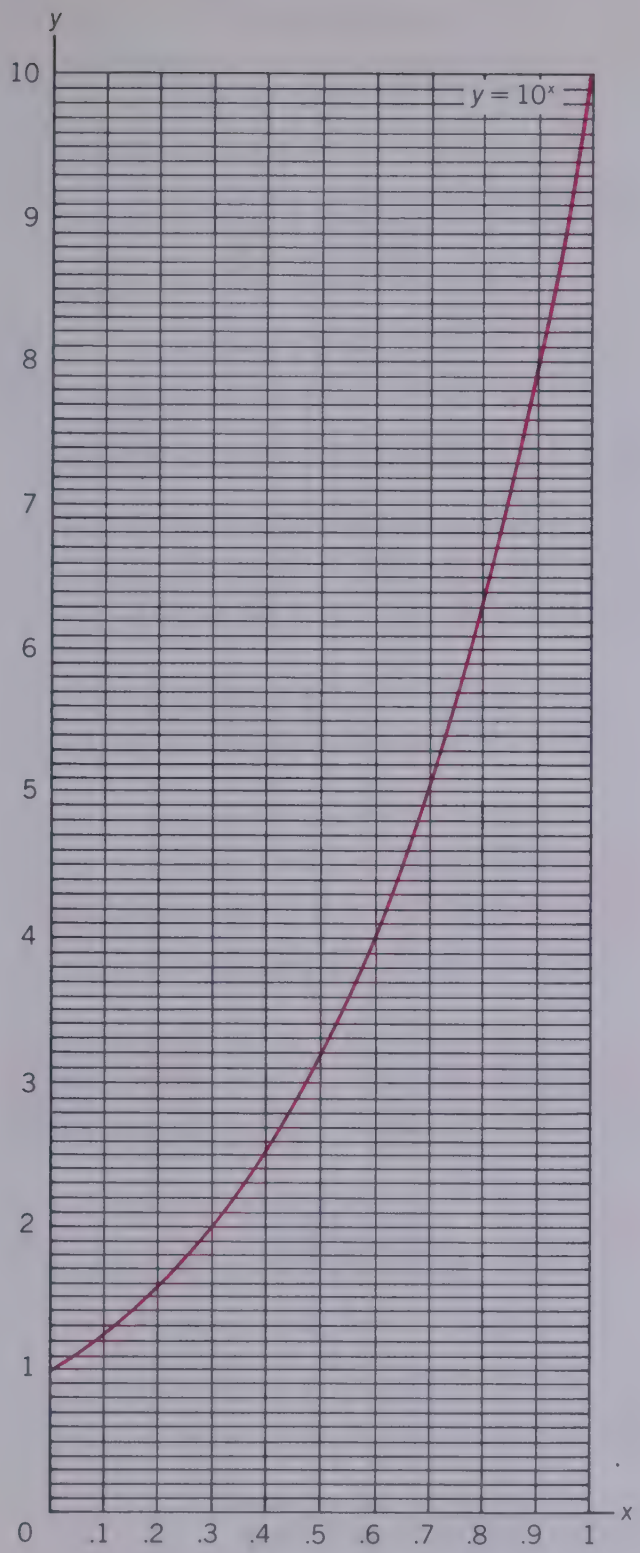


Figure 5-4

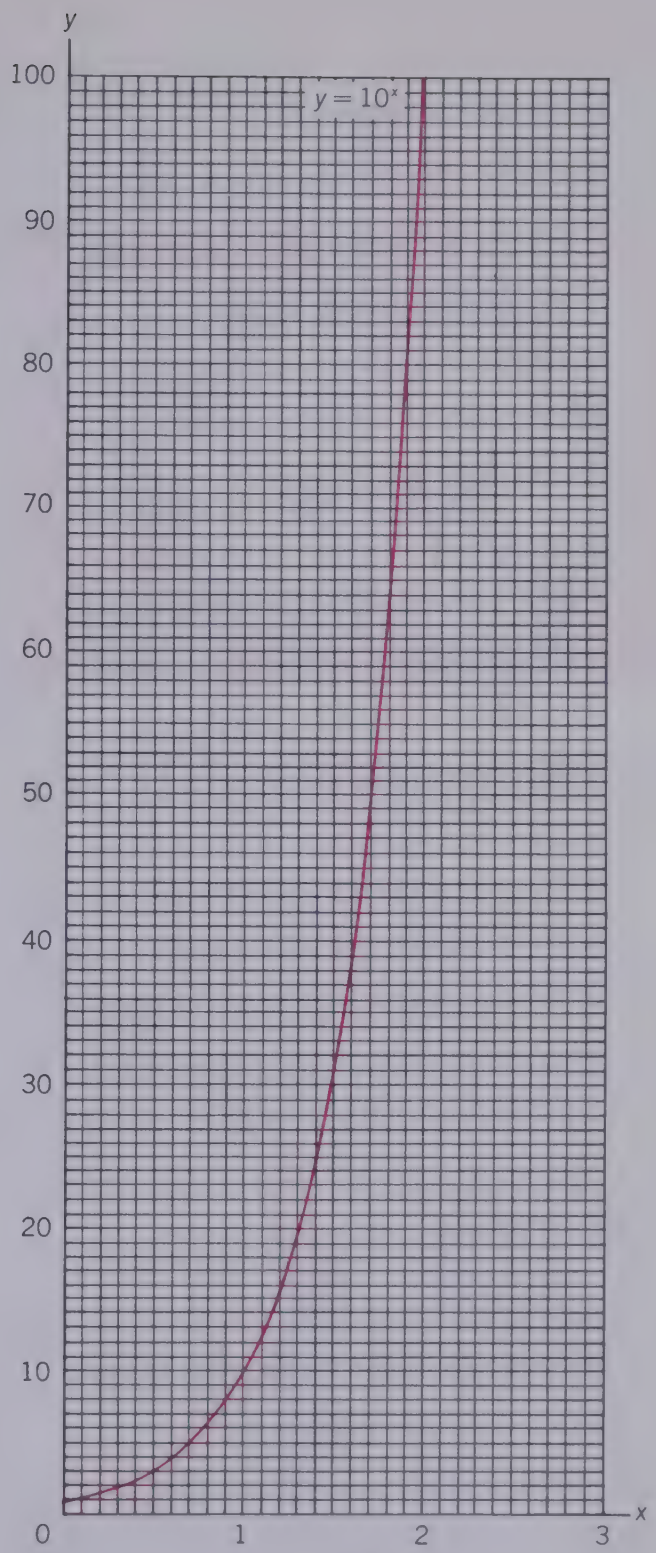


Figure 5-5

EXERCISE 5·8

1. (a) Copy and complete this table. Use the graphs in Figures 5-4 and 5-5.

y	1	2	4	5	7	8	10	20	40	50	70	80
x												

(b) What do you notice about the x values for the first half of the table compared with the second half?

(c) Use the first table of values to complete the second.

y	3	9
x	.47	.95

y	30	90
x		

(d) Express each number given in (a) in scientific notation. What do you notice about the exponent and x values in (c) for corresponding numbers?

2. Write each in exponential form. The first one is completed for you.

(a) $35 = 10^{1.55}$ (b) 3.5 (c) 6.5

(d) 8.5 (e) 5.5 (f) 8.8

(g) 1.5 (h) 43 (i) 56

(j) 38 (k) 63 (l) 78

3. Write each of the numbers in question 2 in logarithmic form. The first one is done for you.

(a) $\log_{10} 35 = 1.55$

4. What is the logarithm of each of these numbers? Use the graphs.

(a) 17 (b) 29 (c) 35 (d) 4.4

(e) 3.9 (f) 5.6 (g) 77 (h) 69

5. What is the mantissa of the logarithm of each of these numbers?

(a) 1.6 (b) 16 (c) 4.2 (d) 42

(e) 7.6 (f) 76

6. What is the characteristic of each number in question 5?

5·9 Using Exponents to Multiply

Remember the *Multiplication Rule for Powers:*

$$10^2 \times 10^3 = 10^5$$
$$10^5 \times 10^4 = 10^9$$
$$10^{1.7} \times 10^{2.4} = 10^{4.1}$$

$$b^a \times b^c = b^{a+c}$$

We shall use this rule and the graphs on page 85 to multiply some simple numbers.

Example 1

Use exponents to multiply:

(a) 12×6 (b) 365×48

Solution

(a) Using the graphs in Figures 5-4 and 5-5 we find

$$12 \simeq 10^{1.07}$$
$$6 \simeq 10^{.78}$$

Therefore
$$12 \times 6 \simeq 10^{1.07} \times 10^{.78}$$
$$\simeq 10^{1.07 + .78} \quad \textit{Why?}$$
$$\simeq 10^{1.85}$$
$$\simeq 10^{.85} \times 10^1$$

Now use the graph again (the opposite way this time).
$$10^{.85} \times 10^1 \simeq 7.2 \times 10^1$$
$$\simeq 72$$

Therefore $12 \times 6 \simeq 72$.

(b) 365×48

Since our graphs do not extend to 365 we shall have to be clever:

$$365 = 3.65 \times 10^2$$
$$48 = 4.8 \times 10^1$$
$$365 \times 48 = (3.65 \times 10^2) \times (4.8 \times 10^1)$$

Using the graphs:
$$\simeq (10^{.56} \times 10^2) \times (10^{.68} \times 10^1)$$
$$\simeq 10^{1.24} \times 10^3 \quad \textit{Why?}$$
$$\simeq 10^{.24} \times 10^4 \quad \textit{Why?}$$

Using the graph again:
$$10^{.24} \times 10^4 \simeq 1.76 \times 10^4$$
$$\simeq 17{,}600$$

Therefore $365 \times 48 \simeq 17{,}600$.

Check this with your slide rule.

Example 2

Use exponents to divide. Remember the power law

$$b^a \div b^c = b^{a-c}$$

(a) $12 \div 3$ (b) $148 \div 64$

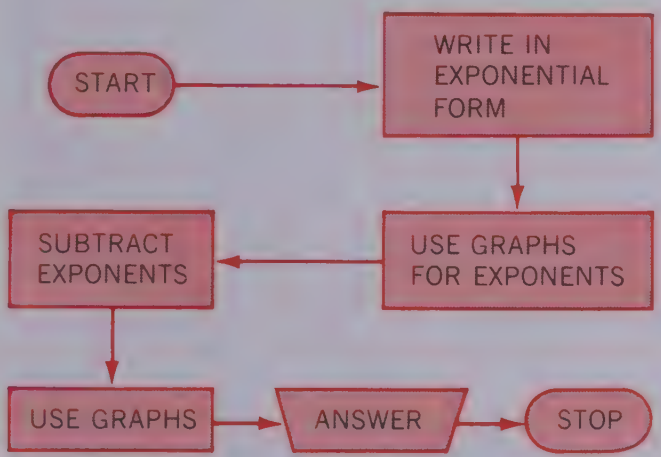
Solution

(a) $12 \div 3 \simeq 10^{1.08} \div 10^{.48}$
$\simeq 10^{1.08 - .48}$
$\simeq 10^{.60}$
$10^{.60} \simeq 4$
$12 \div 3 \simeq 4$

We use the approximately equal sign since *our work is approximate*, even though in this case the answer is obviously 4 exactly.

(b) $448 \div 64 = (4.48 \times 10^2) \div (6.4 \times 10^1)$
$\simeq (10^{.65} \times 10^2) \div (10^{.81} \times 10^1)$
$\simeq 10^{(.65 + 2) - (.81 + 1)}$
$\simeq 10^{.84}$ *Did you follow these?*
$10^{.84} \simeq 7$

Therefore $448 \div 64 \simeq 7$.



EXERCISE 5·9

Use the graphs and exponents to do the following. Use the slide rule to check your answers.

1. Find the products:
   (a) 36×24 (b) 56×40 (c) 74×35
   (d) 150×17 (e) 650×73 (f) 930×6

2. Find the quotients:
   (a) $750 \div 35$ (b) $40 \div 8$ (c) $14.3 \div 7.6$
   (d) $650 \div 82$ (e) $980 \div 6.2$ (f) $460 \div 85$

   Check each by multiplying your answer by the divisor using exponents.

5·10 Finding Logarithms

In expressing the logarithm of a number it is advantageous to first express the number in scientific notation. Then the logarithm takes on the form shown in the following example.

Example 1

Find the logarithm of 753 to base 10.

Solution

$$\log_{10} 753 = \log_{10}(7.53 \times 10^2)$$
$$= 0.88 + 2 \quad \textit{using graphs}$$
$$\text{(mantissa)} \qquad \text{(characteristic)}$$

Note the use of the plus sign. If we use exponential notation

$$753 = 7.53 \times 10^2$$
$$= 10^{0.88} \times 10^2$$
$$= 10^{0.88 \cdot 2}$$

For reasons you will see later it is best to have the exponent expressed in the form shown here: i.e., $0.88 + 2$.

Example 2

Find the $\log_{10} 0.00469$.

Solution

$\log_{10} 0.00469 = \log_{10} (4.69 \times 10^{-3})$
$= 0.67 - 3$ *using graphs*
(mantissa) (characteristic)

Example 3

Find the characteristic of each of the following:

(a) $\log_{10} 3.814$ (c) $\log_{10} 0.00007$
(b) $\log_{10} 4682$ (d) $\log_{10} 0.8005$

Solution

(a) $3.814 = 3.814 \times 10^0$
Therefore the characteristic is 0.

(b) $4682 = 4.682 \times 10^3$
The characteristic is 3.

(c) $0.00007 = 7.0 \times 10^{-5}$
The characteristic is -5.

(d) $0.8005 = 8.005 \times 10^{-1}$
The characteristic is -1.

Population (and its trash) grows exponentially. If Canada's growth rate is 1.4% per year, use logarithms to find the number of years it will take to double the country's population.

EXERCISE 5·10

1. Look up the word mantissa in a dictionary. What does the root word mean? How does this apply to our use in math?

2. Using the graphs find the logarithms of:

   (a) 2.4 (b) 7.3 (c) 5.9
   (d) 9.4 (e) 6.8 (f) 4

3. What is the characteristic of each of the following?

   (a) $\log_{10} 35.72$ (b) $\log_{10} 864$
   (c) $\log_{10} 286500$ (d) $\log_{10} 0.004982$
   (e) $\log_{10} 0.0000093$ (f) $\log_{10} 0.6$

4. Using the graphs find the logarithms of:

   (a) 320 (b) 76.9
   (c) 4390 (d) 62,000
   (e) 845,000 (f) 576,000
   (g) 0.37 (h) 0.00059
   (i) 0.05 (j) 0.0000027

5. Using the graphs find the numbers corresponding to the following logarithms (all to base 10).

   (a) $0.3 + 1$ (b) 0.39
   (c) $0.54 + 3$ (d) $0.84 + 2$
   (e) $0.74 - 1$ (f) $0.69 - 4$
   (g) $0.41 + 1$ (h) $0.17 + 3$
   (i) $0.82 + 7$ (j) $0.62 - 5$

6. Use exponents to calculate the following. Use the graphs on page 85. Check each with your slide rule.

   (a) 560×730 (b) 79×3.8
   (c) 84×5.6 (d) 55×2.4
   (e) $\frac{38 \times 40}{52}$ (f) $\frac{.78 \times .4}{1.39}$
   (g) $\frac{84 \times 52}{13}$ (h) $\frac{870 \times 560}{320}$
   (i) $\frac{710 \times 790}{750}$ (j) $\frac{260 \times 390}{42}$

5·11 The Logarithmic Function

Project

Project 1

Purpose

To find out whether there is a relation between $y = 10^x$ and $x = 10^y$, and if so, what that relation is.

Method

Obtain a very large sheet of graph paper (or tape together smaller sheets).
First, take the defining equation $y = 2^x$ and graph it from the following table (complete the blanks).

x	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$y = 2^x$		$\frac{1}{8}$				2			16			

After locating the coordinate points, join them by a smooth flowing line.

Then take the defining equation $x = 2^y$. Similarly complete the following table of ordered pairs.

y	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$x = 2^y$					1							

On the same set of axes as the graph of $y = 2^x$ plot the coordinate points. Then join them by a smooth flowing line.

What do you observe about the ordered pairs of both tables?

Do the two graphs show any relation to each other?

Now draw a straight line defined by $y = x$ on the same axes as $y = 2^x$ and $x = 2^y$.

Does this line help you make a further observation about the two graphs, $y = 2^x$ and $x = 2^y$.

Compare your graphs to those of Figure 5-5. These graphs of $y = 10^x$ and $x = 10^y$ are similar to the ones of $y = 2^x$ and $x = 2^y$.

Questions

1. Are graphs of $y = 2^x$ and $x = 2^y$ symmetric? If so, with respect to what line?

2. Do graphs of $y = 10^x$ and $x = 10^y$ exhibit the same symmetry?

3. What do you observe about the first elements of the ordered pairs for $y = 2^x$ and the second elements of the ordered pairs for $x = 2^y$? What do you observe about second elements of the ordered pairs for $y = 2^x$ and first elements of the ordered pairs for $x = 2^y$? Do you believe there is a relation between this observation and the one on symmetry?

When first elements of ordered pairs for a function become second elements of the ordered pairs of another function (and vice versa), then the domain of one function is the range of the other function (and vice versa). In cases like these, if both relations are functions, they are called inverse functions. Inverse functions are always symmetric with respect to $y = x$.

Since $x = 10^y$ can be stated equivalently as $\log_{10} x = y$, i.e., $y = \log_{10} x$, then we may conclude that $y = 10^x$ and $y = \log_{10} x$ are defining equations of inverse functions.

Are calculations with logarithms exact? Is there such a thing as an exact measurement?

Example 1

What are the defining sentences of the inverse functions of:

(a) $y = 5^x$ (b) $y = \log_7 x$

Solution

(a) For $y = 5^x$ the inverse is $x = 5^y$ (by replacing the y with an x and the x with a y).

Since $x = 5^y$ is equivalent to $y = \log_5 x$, then the inverse of $y = 5^x$ is $y = \log_5 x$.

(b) Similarly for $y = \log_7 x$ the inverse is $x = \log_7 y$ i.e., $y = x^7$.

EXERCISE 5·11

1. Write the inverses of each of the following in logarithmic form.
   (a) $y = 4^x$ (b) $y = 9^x$ (c) $y = b^x$
2. Write the inverses of each of the following in exponential form.
   (a) $y = \log_3 x$ (b) $y = \log_8 x$
   (c) $y = \log_m x$
3. Use the graph in Figure 5-6 to find the value of x in each of:
   (a) $\log_{10} x = 0.3$ (b) $\log_{10} x = 0.7$
   (c) $\log_{10} x = .3 + 1$ (d) $\log_{10} x = .65 + 2$.

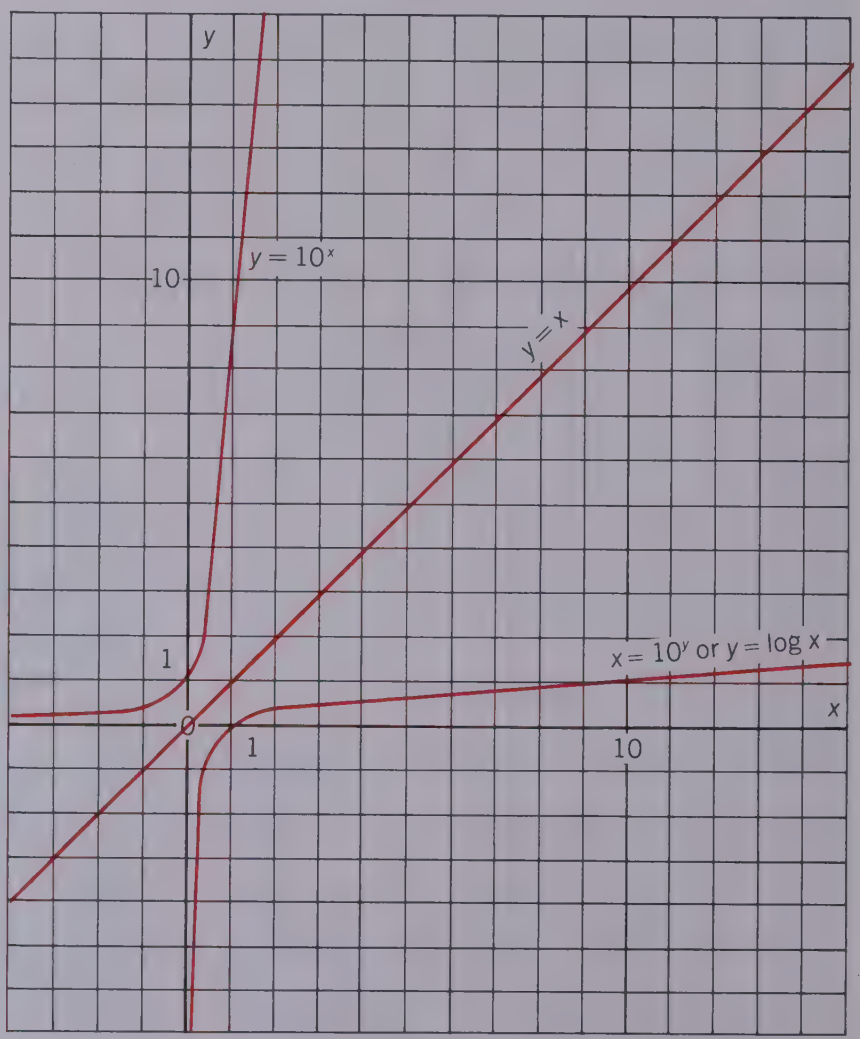


Figure 5-6

5·12 Logarithm and Antilogarithm Tables

Have you experienced difficulty being very accurate when using the graphs for exponents and the inverses? Mathematicians have developed tables for $y = 10^x$ and these tables are called *Logarithm Tables*. The following is a portion of a "log table". Note only the mantissa is given. Also no decimal is shown. It is understood that a decimal precedes each number.

	0	4	6	7	8	9	Mean Differences 1 2 3	Mean Differences 4 5
17	23045	24055					26 50 76	101 126
			24551	24797	25042	25285	25 49 73	98 122
18	25527	26482					24 48 71	95 119
			26951	27184	27416	27646	23 46 69	93 116
19	27875	28780					23 45 68	90 113
			29226	29447	29667	29885	22 44 66	88 110
20	30103	30963	31387	31597	31806	32015	21 43 64	85 106
21	32222	33041	33445	33646	33846	34044	20 40 60	80 100
22	34242	35025	35411	35603	35793	35984	20 39 58	77 97
23	36173	36922	37291	37475	37658	37840	19 37 56	74 93
24	38021	38739	39094	39270	39445	39620	18 35 53	71 89
25	39794	40483	40824	40993	41162	41330	17 34 51	68 85
26	41497	42160	42488	42651	42813	42975	16 33 49	66 82
27	43136	43775	44091	44248	44404	44560	16 32 47	63 79
28	44716	45332	45637	45788	45939	46090	15 30 46	61 76
29	46240	46835	47129	47276	47422	47567	15 29 44	59 74
30	47712	48287	48572	48714	48855	48996	14 29 43	57 72
31	49136	49693	49969	50106	50243	50379	14 28 41	55 69
32	50515	51054	51322	51455	51587	51720	13 27 40	54 67
33	51851	52375	52634	52763	52892	53020	13 26 39	52 65
34	53148	53656	53908	54033	54158	54283	13 25 38	50 63
35	54407	54900	55145	55267	55388	55509	12 24 37	49 61
36	55630	56110	56348	56467	56585	56703	12 24 36	48 60

Note! When log x is written with no base, it is understood the base is 10.

$$\log 354 \approx \log (3.54 \times 10^2)$$
$$\approx .5490 + 2$$

$$\log 3274 = \log (3.274 \times 10^3)$$
$$\approx .5150 + 3 \qquad 5145 + 5 = 5150$$

Do you see that you have to *add* the *mean difference* if you have four significant figures? If you have more than four significant figures, round the number to four figures.

To assist you in finding the inverse we have antilogarithm tables. They are used in the same way the *Log Tables* are used.

	0	1	2	3	5	Mean Differences 4 5 6	Mean Differences 7 8 9
.15	14125	14158	14191	14223	14289	13 16 20	23 26 30
.16	14454	14488	14521	14555	14622	13 17 20	24 27 30
.17	14791	14825	14859	14894	14962	14 17 21	24 28 31
.18	15136	15171	15205	15241	15311	14 18 21	25 28 32
.19	15488	15524	15560	15596	15668	14 18 22	25 29 32
.20	15849	15885	15922	15959	16032	15 18 22	26 30 33
.21	16218	16255	16293	16331	16406	15 19 23	26 30 34
.22	16596	16634	16672	16711	16788	15 19 23	27 31 35
.23	16982	17022	17061	17100	17179	16 20 24	28 32 36
.24	17378	17418	17458	17498	17579	16 20 24	28 32 36
.25	17783	17824	17865	17906	17989	17 21 25	29 33 37
.26	18197	18239	18281	18323	18408	17 21 25	30 34 38
.27	18621	18664	18707	18750	18836	17 22 26	30 35 39
.28	19055	19099	19143	19187	19275	18 22 26	31 35 40
.29	19498	19543	19588	19634	19724	18 23 27	32 36 41

$$\text{antilog } (0.173 + 1) \approx 1.489 \times 10^1 \qquad \textit{Did you follow?}$$
$$\approx 14.89$$

$$\text{antilog } (0.1959 - 2) \approx 1.570 \times 10^{-2}$$
$$1567 + 3 = 1570$$
$$\approx 0.0157$$

Note! We can find the antilogs by using the log tables.

	7	8	9	Mean Differences 1 2 3	Mean Differences 4 5 6	Mean Differences 7 8 9
12	10380	10721	11059	36 71 106 34 68 102	142 177 213 136 170 204	248 284 319 238 272 307
13	13672	13988	14301	33 66 98 32 63 95	131 164 197 126 158 190	229 262 295 221 253 284
14	16732	17026	17319	30 61 91 29 59 88	122 152 183 118 147 177	213 244 274 206 236 265
15	19590	19866	20140	28 57 85 28 55 83	114 142 171 110 138 165	199 228 256 193 221 248
16	22272	22531	22789	27 53 80 26 52 78	107 134 160 104 130 156	187 214 240 182 208 233

We look in the table for 173. The closest is 1732. We look to the left column and see 14, then look to the top row and see 9. The antilog is 149.

$$\text{antilog } 0.173 \approx 1.49$$

Similarly antilog 0.1959 is found by looking for 1959 in the table. We look left and up and get

$$\text{antilog } 0.1959 \approx 1.57.$$

Example 1

Use logarithms and tables to calculate 156×74.

Solution

Let $x \simeq 156 \times 74$

$x \simeq 1.56 \times 10^2 \times 7.4 \times 10^1$

$\log x \simeq \log (1.56 \times 10^2) + \log (7.4 \times 10^1)$

$\simeq 0.1931 + 2 + 0.8692 + 1$ *Log Tables*

$\simeq 1.0623 + 3$

$\simeq 0.0623 + 4$ *Note how we keep the mantissa less*

$x \simeq \text{antilog } (0.0623 + 4)$ *than 1.*

$\simeq 1.154 \times 10^4$ *Antilog tables*

$\simeq 11540$

Since our least accurate figure (74) has only two significant figures we should round our answer to two significant figures, hence

$$156 \times 74 \simeq 12{,}000.$$

Check this with your slide rule.

EXERCISE 5·12

Use tables for the following.

1. Find the logarithm of each:

(a) 33	(b) 345	(c) 856
(d) 769	(e) 84.35	(f) 76.325
(g) 69.414	(h) 76.438	(i) 0.03154
(j) 0.76147	(k) 0.0417	(l) 0.006

2. Find the antilogarithm of each:

(a) $0.237 + 1$	(b) $0.7145 + 2$	(c) $0.3415 + 3$
(d) $0.3214 - 2$	(e) $0.4414 + 3$	(f) $0.5768 - 2$
(g) $0.5149 - 1$	(h) $0.4893 - 1$	(i) $0.5678 + 2$
(j) $0.7643 - 4$	(k) $0.9444 - 1$	(l) $0.4774 + 4$

3. Use logs to solve:

(a) 540×47	(b) 72×36	(c) 145×273
(d) 42×8	(e) 1781×5489	(f) 1634×4114

5·13 Computations Using Logarithms

Now that we have established that the logarithm of a number is the exponent of the power of that number, we can proceed to do computations using logarithms. The power laws of Section 5·5 apply.

From $b^a \times b^c = b^{a+c}$ where the exponents are added in the multiplication rule, we obtain the following:

Multiplication Rule for Logarithms

> If $M, N \in {}^+R$, then
> $\log MN = \log M + \log N$

Example 1

Multiply 735.4 by 0.48.

Solution

Let $x = 735.4 \times 0.48$

$= (7.354 \times 10^2) \times (4.8 \times 10^{-1})$

$\log x = \log (7.354 \times 10^2) + \log (4.8 \times 10^{-1})$

$\simeq (0.8665 + 2) + (0.6812 - 1)$

$\simeq 1.5477 + 1$

$\simeq 0.5477 + 2$

$x \simeq 3.530 \times 10^2$

Round to two digits: $x = 350$.

Division Rule for Logarithms

> If $M, N \in {}^+R$, then
> $\log \frac{M}{N} = \log M - \log N$

$$\log \frac{M}{N} = \log M - \log N$$

Example 2

Divide 5746 by 84.2.

Solution

Let $x = 5746 \div 84.2$

$= (5.746 \times 10^3) \div (8.42 \times 10^1)$

$\log x = \log (5.746 \times 10^3) - \log (8.42 \times 10^1)$

$\simeq (0.7594 + 3) - (0.9253 + 1)$

$\simeq (1.7594 + 2) - (0.9253 + 1)$

Make adjustment to subtract.

$\simeq (1.7594 + 2) - 0.9253 - 1$

$\simeq 0.8341 + 1$

$\text{antilog } (0.8341 + 1) \simeq x$

$x \simeq 6.825 \times 10^1$

Round to three digits: $x \simeq 683$. *Why?*

It is to be noted that the mantissa is *always* to be *positive*. If we subtract 0.9253 from 0.7594 the answer is negative. This is not allowed in logarithms. Therefore we must take 1 from the characteristic 3 reducing it to 2, and add this 1 to the mantissa 0.7594 to get 1.7594. We can now subtract and get a positive mantissa.

$$1.7594 - 0.9253 = 0.8341$$

You must always be careful to avoid negative mantissas.

Rule Always use positive mantissas.

Example 3

Calculate $\dfrac{5130 \times 0.084}{0.0067}$.

Solution

Let $x = \dfrac{(5.13 \times 10^3) \times (8.4 \times 10^{-2})}{6.7 \times 10^{-3}}$

$\log x = \log (5.13 \times 10^3) + \log (8.4 \times 10^{-2}) - \log (6.7 \times 10^{-3})$

$\simeq (0.7101 + 3) + (0.9243 - 2) - (0.8261 - 3)$

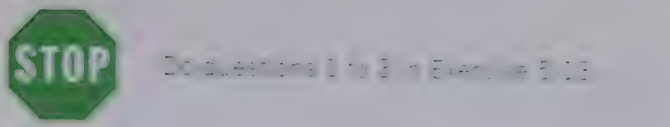
$\simeq (1.6344 + 1) - (0.8261 - 3)$

$\simeq 0.8083 + 4$

$\text{antilog } (0.8083 + 4) \simeq x$

$x \simeq 6.431 \times 10^4$

Round to two significant figures: $x = 64.000$.



Do questions 1 to 3 in Exercise 5.13

....The computer then selects the correct equation from the pattern of theoretical data, utilizes the most probable replacements for the variables, performs the complicated operations, calculates the answer and prints it on a card. Then our Miss Olsen files it where it will never be found again.

Powers and Roots Rule for Logarithms

If $M \in {}^+R$ and $K \in R$, then
$\log M^K = K \log M$

Example 4

Calculate 42^5.

Solution

Let $x = 42^5$.

$\log x = 5 \log 42$

$= 5 \log (4.2 \times 10^1)$

$\simeq 5 (0.6232 + 1)$

$\simeq 3.1160 + 5$

$\simeq 0.1160 + 8$

$\text{antilog } (0.1160 + 8) = x$

$x \simeq 1.3 \times 10^8$ *Why?*

Example 5

Calculate $\sqrt[3]{0.072}$.

Solution

Let $x = \sqrt[3]{0.072}$

$= (7.2 \times 10^{-2})^{\frac{1}{3}}$

$\log x = \frac{1}{3} \log (7.2 \times 10^{-2})$

$\simeq \frac{1}{3}(0.8573 - 2)$

$\simeq \frac{1}{3}(1.8573 - 3)$ *Make adjustment to*

$\simeq 0.6191 - 1$ *multiply by* $\frac{1}{3}$.

$x \simeq \text{antilog}\,(0.6191 - 1)$

$x \simeq 4.160 \times 10^{-1}$

$x \simeq 4.2 \times 10^{-1}$ *Why?*

Note! We must make another tricky adjustment here. The *characteristic must always be an integer.* Since $2 \div 3$ is not an integer we shall have to add to it (or subtract from it) to make it divisible by 3. In this case we subtract 1 from -2 and get -3. Now since we subtracted 1 from $(0.8573 - 2)$ to get $(0.8573 - 3)$ this is no longer equivalent to what we had before. Therefore we will add 1 to 0.8573 to get $(1.8573 - 3)$. Now $(0.8573 - 2)$ is equivalent to $(1.8573 - 3)$. When $(1.8573 - 3)$ is divided by 3 the characteristic is an integer.

Rule

The characteristic *must* always be an integer.

Example 6

Calculate $\dfrac{(425)^2\,(\sqrt[3]{896})}{\sqrt{0.066}}$.

Solution

Let $x = \dfrac{(4.25 \times 10^2)^2 \times (8.96 \times 10^2)^{\frac{1}{3}}}{(6.6 \times 10^{-2})^{\frac{1}{2}}}$

$\log x = 2 \log (4.25 \times 10^2) + \frac{1}{3} \log (8.96 \times 10^2) - \frac{1}{2} \log (6.6 \times 10^{-2})$

$\log x \simeq 2\,(0.6284 + 2) + \frac{1}{3}(0.9523 + 2) - \frac{1}{2}(0.8195 - 2)$

$\simeq (1.2568 + 4) + \frac{1}{3}(2.9523) - (0.4098 - 1)$

$\simeq (1.2568 + 4) + (0.9841) - (0.4098 - 1)$

$\simeq 0.8311 + 6$

$x = \text{antilog}\,(0.8311 + 6)$

$x \simeq 6.778 \times 10^6$

$x \simeq 6.8 \times 10^6$ *Why?*

EXERCISE 5·13

1. Explain in your own words the reason for the
   (a) multiplication rule for logarithms
   (b) division rule for logarithms.

2. By using logarithms, calculate:

(a) 35×72	(b) 480×8200
(c) 0.0055×367	(d) 0.42×0.32
(e) 740×0.73	(f) $8.4 \times .85$
(g) 76×76	(h) $44 \times 44 \times 44$
(i) $840 \div 76$	(j) $760 \div 52$
(k) $492 \div 37$	(l) $6780 \div 0.053$
(m) $7640 \div 5.3$	(n) $0.04 \div 76$

3. Use logarithms to calculate:

(a) $\frac{360 \times 72}{56}$	(b) $\frac{830 \times 42}{95}$
(c) $\frac{71 \times 64}{87}$	(d) $\frac{114 \times 173}{564}$
(e) $\frac{763 \times 564}{73 \times 56}$	(f) $\frac{341 \times 584}{974 \times 0.3}$

4. Explain in your own words the reason for the powers and roots rule for logarithms. Refer to the power rule for exponents.

5. Use logarithms to calculate:

(a) 85^2	(b) 0.246^2	(c) 3.85^3
(d) $\sqrt{240}$	(e) $\sqrt{685}$	(f) $\sqrt[3]{5640}$
(g) $\sqrt{81}$	(h) $\sqrt{810}$	(i) $\sqrt{640}$

Calculate the following:

6. 4917×28.63

7. 926.72×0.00077

8. $\dfrac{0.01156 \times 38{,}269}{477.6}$

9. $\dfrac{53.8^2 \times 0.291^3}{4.71^4 \times 0.0036}$

10. 3.79^5

11. $(0.8846)^4$

12. $\sqrt{37.560}$

13. $\sqrt[3]{29.66}$

14. $\sqrt[4]{0.02576}$

15. $\dfrac{(3.7)^5 \times \sqrt[3]{29.460}}{\sqrt{4893}}$

16. $\sqrt{\dfrac{0.9117 \times 5.32}{0.00062}}$

17. $\sqrt[4]{\dfrac{9.37}{0.865^3}}$

18. $\sqrt[5]{8893^3} \times \sqrt[3]{2.16^4}$

19. $\sqrt[3]{59.46^2}$

Using the Library

1. Prepare a time line for the period indicated by the lives of the following people and the events. Also label the time line:

   Archimedes – value of π, How was he killed?
   Apollonius – cut a cone to form conic sections.
   Hipparchus – beginning of trigonometry.
   Julius Caesar – new calendar
   Romans invade Britain
   Birth of Christ
   Heron of Alexandria – formula
   Menelaus – Theorem
   Hadrian's Wall – Why was it built?
   Ptolemy – Almagest
   Emperor Shi Huang-ti of Ch'in Dynasty – Burn all books

2. Describe the political climate in which Archimedes lived. How was Archimedes involved in the political problems? What were some of Archimedes' inventions that were a result of the involvement?

3. Julius Caesar was a noteworthy mathematician as well as a great statesman. Describe some of his mathematical accomplishments.

4. Emperor Shi Huang-ti of the Ch'in Dynasty was not the only ruler to burn books. What were the causes and the affect of the burning? Discuss the reasons a ruler may decree that all books be burned. What were the punishments for disobeying the decree? Read *Fahrenheit 451* by Ray Bradbury (published by Macmillan of Canada). Discuss the message Bradbury is trying to get across.

5. Refer to *An Introduction to the History of Mathematics* page 138. Discuss in groups the questions in Section 5-14. Which is best, intuition or proof?

6. What was the "Sieve of Eratosthenes"? Make the Sieve to find the first 100 positive integers.

REVIEW EXERCISE 5

1. Simplify:
   (a) $\dfrac{\sqrt{6}}{3+\sqrt{3}}$ (b) $\dfrac{5\sqrt{8}-6\sqrt{2}}{3\sqrt{2}+\sqrt{8}}$

2. Solve for n:
   (a) $\sqrt{4n+13} = 2+\sqrt{3n}$
   (b) $\sqrt{4n+4}-\sqrt{n+1} = 3$

3. Simplify:
   (a) $\dfrac{5^6 \times 25^4}{125^5}$ (b) $\dfrac{27^{7n} \times 9^{3n}}{81^{5n}}$
   (c) $64^{\frac{5}{6}} \times 8^{-\frac{4}{3}} \times 16^{\frac{3}{4}} \times 32^{-\frac{4}{5}}$

4. Use logarithms to compute:
   (a) 4732×758 (b) $9625 \div 38.4$
   (c) $\sqrt[3]{(2.84)^2}$ (d) $\dfrac{89.17 \times 0.45}{0.0076}$

Your Job and Mathematics

Automotive Mechanics
Bob & John Watt
Memorial Drive & Louise Ave.

People bring all types of problems to mechanics. It is expected that only problems related to automobiles would appear at the shop but consider these problems below as several that we get periodically.

1. An excavating company brings a water pump engine in for repair. The claim is that it will not pump the water it should. Fix it.

   On inspection we note the required speed for the pump is 550 revolutions per minute. The gasoline engine runs at 1350 revolutions per minute. The motor has a $2\frac{1}{2}$-inch pulley and the pump has a $7\frac{1}{2}$-inch pulley. If the motor can gain full speed will the pump be at its recommended speed? If not, what diameter should the pulley be on the pump?

2. The mechanic must also calculate to see if the motor has enough horsepower at normal running speed, to run the pump. The pump specifications state a 6.5 horsepower gasoline motor is required to operate the pump efficiently. The mechanic recalls a general formula that gives the approximate horsepower (*H.P.*) of gasoline engines:

   $$H.P. = \frac{D^2N}{2.5}$$

   where D is the diameter of each piston in inches, and N is the number of pistons.

   The motor in question has two pistons with diameters of 2.4 inches each.

   What is the approximate horsepower of the motor if it is running well?

   If you were the mechanic what would you tell the excavating company about the pumping system?

Project

Project 1

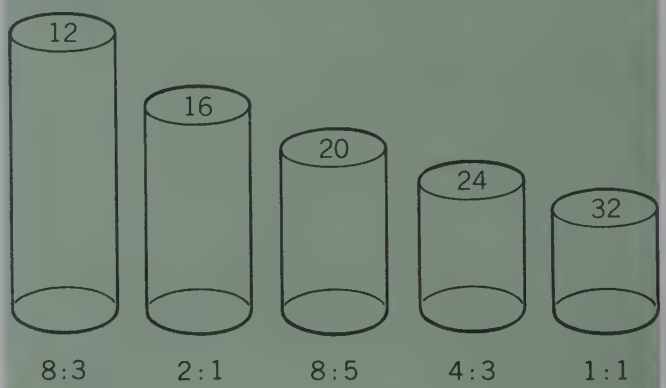
This project should only be done in a strictly supervised area where extreme safety precautions can be exercised.

Purpose

1. To check the rate of acceleration of a car (or a motorcycle or a bicycle).
2. To check the brakes on the vehicle used in part 1.

Method

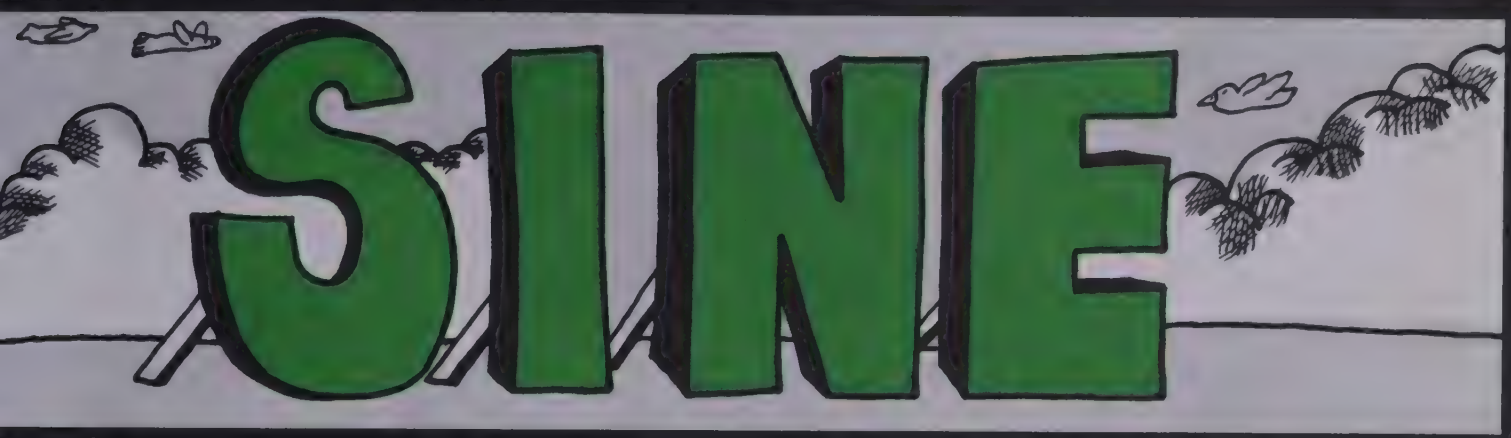
1. Prepare the set of cylindrical blocks shown. The ratio given under each is the height of the block to its diameter. Print the numerals (mph) given on the top of the blocks as shown.

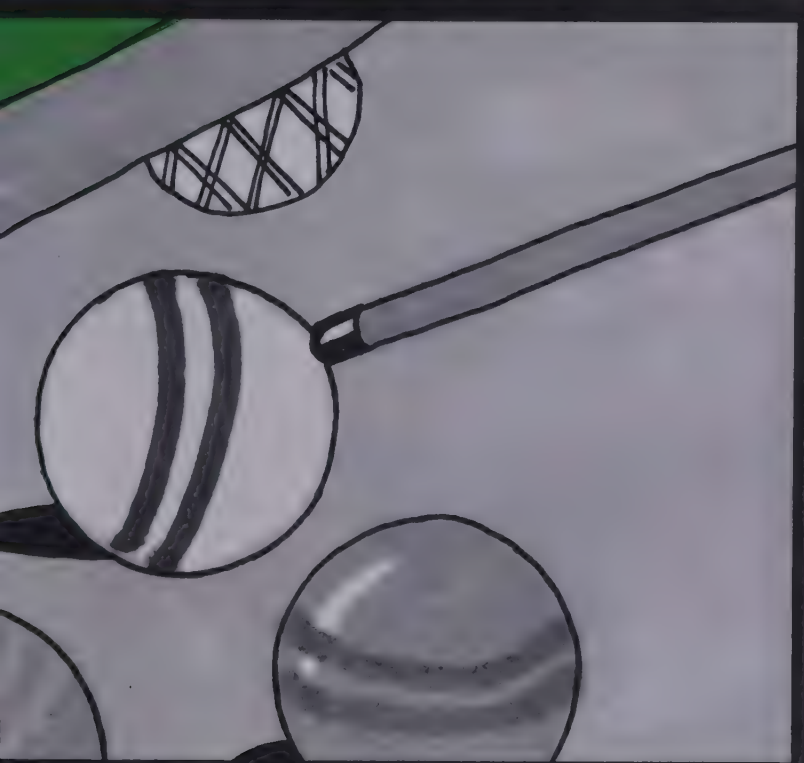


2. Place the blocks upright on end on a level portion of the floor of your car. (For a motorcycle you will have to rig up an open top box the bottom of which is level. Then place the blocks on end on the bottom of the box.)
3. Start the motor and accelerate smoothly as fast as possible. After a period of acceleration, slow down and stop slowly. The block with the highest numeral that falls gives the acceleration reached.
4. Placing the blocks upright again accelerate slowly to 20 mph. Then apply the brakes and decelerate as rapidly as possible. The block with the highest numeral that tips over indicates the braking rate or deceleration. Good brakes mean at least the 20 block should tip.

Chapter Six

TRIGONOMETRY





6·1 Primary Trigonometric Functions

If you have studied the section on trigonometry in **Math Probe 2** or **3**, you may wish to proceed directly to Exercise 6.2 for review purposes. Then proceed with Section 6.3 which is new.

Trigonometry is a branch of mathematics concerned primarily with relations of angles and sides of a triangle. It is extensively used in surveying, construction, navigation and astronomy.

Let us first examine several right triangles.

Project

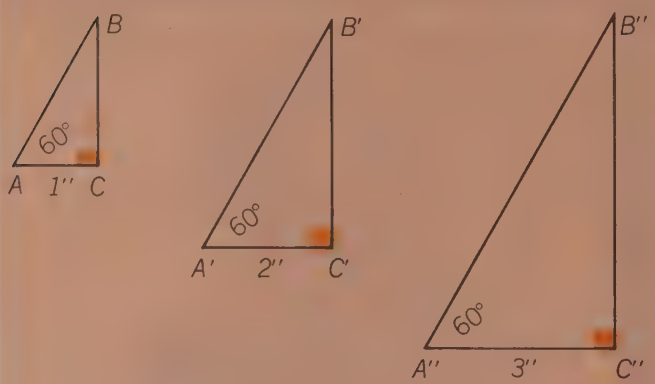
Project 1

Purpose

To discover the relation between ratios of sides of a right triangle and the measure of the one acute angle.

Method

1. As accurately as possible, construct the following triangles:



2. Use a ruler to find the following ratios of measures:

$\frac{m(\overline{AC})}{m(\overline{AB})}$ $\frac{m(\overline{BC})}{m(\overline{AB})}$ $\frac{m(\overline{BC})}{m(\overline{AC})}$

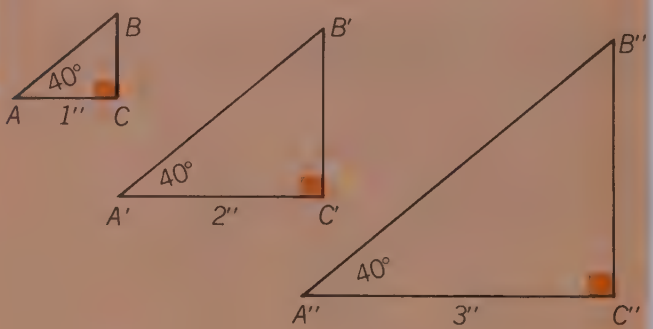
$\frac{m(\overline{A'C'})}{m(\overline{A'B'})}$ $\frac{m(\overline{B'C'})}{m(\overline{A'B'})}$ $\frac{m(\overline{B'C'})}{m(\overline{A'C'})}$

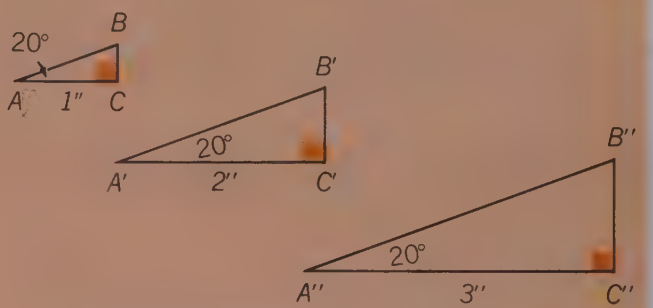
$\frac{m(\overline{A''C''})}{m(\overline{A''B''})}$ $\frac{m(\overline{B''C''})}{m(\overline{A''B''})}$ $\frac{m(\overline{B''C''})}{m(\overline{A''C''})}$

(a) What do you observe about the triangles?

(b) What do you observe about the ratios?

3. Repeat steps 1 and 2 for these triangles.





(a) What do you observe about the ratios?

(b) What do you observe about the ratios $\frac{m(\overline{AC})}{m(\overline{AB})}$ for the three different cases? about $\frac{m(\overline{BC})}{m(\overline{AB})}$? about $\frac{m(\overline{BC})}{m(\overline{AC})}$?

(c) Write a general statement about what happens to each of these ratios as the angle increases in measure.

$\frac{m(\overline{AC})}{m(\overline{AB})}$ $\frac{m(\overline{BC})}{m(\overline{AB})}$ $\frac{m(\overline{BC})}{m(\overline{AC})}$

One of the conclusions you may have drawn is that the sizes of the triangles do not in any way affect the ratio of the measures of two corresponding sides provided the measures of the angles do not change. Also, the ratio of the measures of two sides of a triangle is a function of the measure of one acute angle of a right triangle.

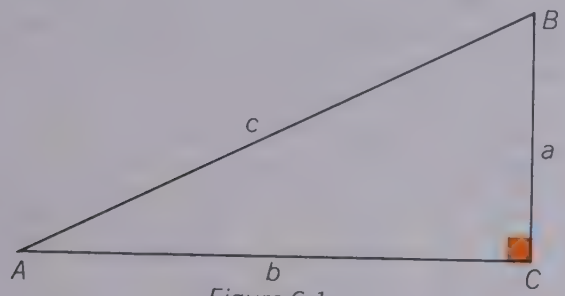


Figure 6-1

Consider the above right triangle ABC. We will now name the functions of the angle A and give the corresponding ratio for each.

sine $A = \frac{a}{c}$, cosine $A = \frac{b}{c}$, tangent $A = \frac{a}{b}$

In abbreviated form they appear as follows:

$\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$

Note! sin A means sine of angle A, i.e., A means $\angle A$ or $m\angle A$.

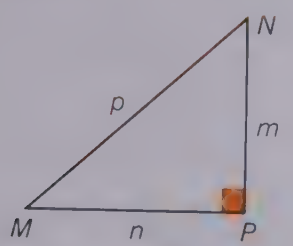


Figure 6-2

In a right triangle, MNP, Figure 6-2, where P is the right angle

$$\sin M = \frac{m}{p} = \frac{\text{side opposite to } \angle M}{\text{hypotenuse}}$$

$$\cos M = \frac{n}{p} = \frac{\text{side adjacent to } \angle M}{\text{hypotenuse}}$$

$$\tan M = \frac{m}{n} = \frac{\text{side opposite to } \angle M}{\text{side adjacent to } \angle M}$$

Similarly:

$$\sin N = \frac{n}{p} = \frac{\text{side opposite to } \angle N}{\text{hypotenuse}}$$

$$\cos N = \frac{m}{p} = \frac{\text{side adjacent to } \angle N}{\text{hypotenuse}}$$

$$\tan N = \frac{n}{m} = \frac{\text{side opposite to } \angle N}{\text{side adjacent to } \angle N}$$

Example 1

For $\triangle ABC$ find sin A, tan A, cos B and tan B.

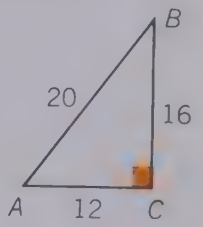
Solution

$\sin A = \frac{16}{20} = \frac{4}{5}$ or .8

$\tan A = \frac{16}{12} = \frac{4}{3} = 1.3\dot{3}$

$\cos B = \frac{16}{20} = \frac{4}{5} = .8$

$\tan B = \frac{12}{16} = \frac{3}{4} = .75$



Example 2

Given $\sin A = \frac{3}{7}$ find cos A and tan A.

Solution

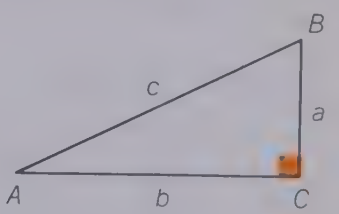
Draw a right triangle ABC with C the right angle. Now since $\sin A = \frac{3}{7}$ we know the side opposite A is 3 and the hypotenuse is 7. Label. Using the Theorem of Pythagoras we can find the adjacent side AC.

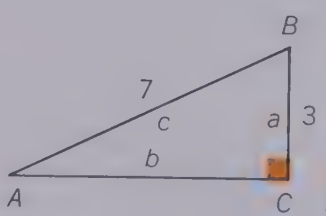
$b = \sqrt{c^2 - a^2}$

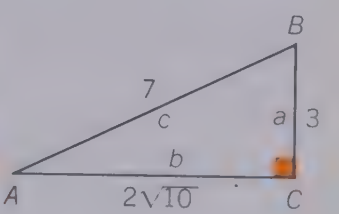
$= \sqrt{49 - 9}$

$= \sqrt{40}$

$b = 2\sqrt{10}$



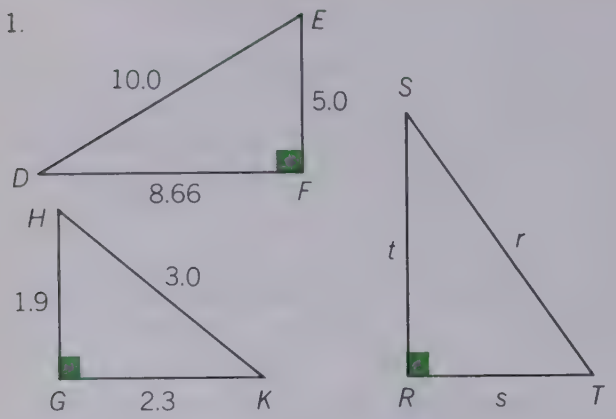




Therefore $\cos A = \frac{2\sqrt{10}}{7}$ and $\tan A = \frac{3}{2\sqrt{10}}$.

EXERCISE 6·1

1.



From the above triangles determine each of the following:

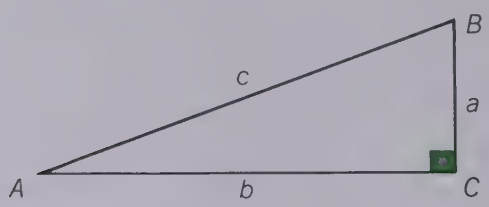
(i) cos D (ii) tan E (iii) cos E

(iv) sin K (v) cos K (vi) tan H

(vii) sin T (viii) tan T (ix) sin S

(x) tan D (xi) sin H (xii) tan S

2. For a right triangle with side c as the hypotenuse, the Pythagorean relation is $c^2 = a^2 + b^2$.



(a) If $\sin A = \frac{103}{200}$, find $\cos A$ (approximate).

(b) If $\cos B = \frac{\sqrt{2}}{2}$, find $\tan B$.

(c) If $\sin B = \frac{\sqrt{3}}{2}$, find $\cos B$.

(d) If $\tan A = 1\frac{1}{3}$, find $\sin A$.

(e) If $\cos B = \frac{\sqrt{2}}{10}$, find $\tan B$.

sin θ

6·2 Using Trigonometric Tables

Since the ratio of two sides of a right triangle is a trigonometric function of one of the angles, a table of values of the trigonometric functions has been computed. The following is a small part of one such table.

Angle Degrees	Sine	Cosine	Tangent
45°	.7071	.7071	1.0000
46°	.7193	.6947	1.0355
47°	.7314	.6820	1.0724
48°	.7431	.6691	1.1106
49°	.7547	.6561	1.1504
50°	.7660	.6428	1.1918
51°	.7771	.6293	1.2349
52°	.7880	.6157	1.2799
53°	.7986	.6018	1.3270
54°	.8090	.5878	1.3764
55°	.8192	.5736	1.4281
56°	.8290	.5592	1.4826
57°	.8387	.5446	1.5399
58°	.8480	.5299	1.6003
59°	.8572	.5150	1.6643
60°	.8660	.5000	1.7321
61°	.8746	.4848	1.8040

So, if we take the triangle ABC as illustrated:

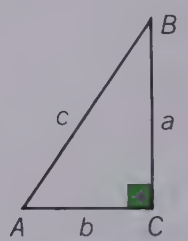


Figure 6-3

and, taking $A = 52°$

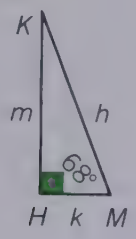
$\sin 52° = \frac{a}{c} = .7880$ *from above table*

$\cos 52° = \frac{b}{c} = .6157$ *from above table*

$\tan 52° = \frac{a}{b} = 1.2799$ *from above table*

Example 1

An engineer surveyed a land location. He presented the accompanying diagram.



In triangle KHM, angle H is a right angle, $m\angle M = 68°$, and $h = 30$.

(a) Calculate k and m.

(b) A survey of a similar lot showed $m = 15$; $h = 21$ and $m\angle H = 90°$. Calculate angles M and K.

Solution

(a) Since we know $m\angle M$ and side h and we need to find side k, we choose a function in which all three are found.

That function is $\cos M = \frac{k}{h}$.

By substitution $\cos 68° = \frac{k}{30}$.

From tables $.3746 = \frac{k}{30}$.

$\therefore k = 30(.3746) = 11.238$.

To 2 significant figures $k = 11$.

Now we know side h, $\angle M$ and want to find side m.

We use the relation, $\sin M = \frac{m}{h}$.

By substitution $\sin 68° = \frac{m}{30}$.

From tables $.9297 = \frac{m}{30}$.

$\therefore m = 30(.9297) = 27.891$

To 2 significant figures $m = 28$.

(b) To find $\angle M$ when we know m and h, we choose

$$\sin M = \frac{m}{h}$$

By substitution $\sin M = \frac{15}{20} = .7143$.

We now look for .7143 in the sin column. We note it comes between .7071 and .7193. That is

$$45° < m\angle M < 46°$$

It appears to be closer to 46°. Can you explain why?

We can either estimate the fractional degree (it is close to $45\frac{1}{2}$) or by use of more accurate tables we can find the $m\angle M$ to a greater degree of accuracy. At this time we will round off to what appears to be the nearest whole degree.

To find $\angle K$, we can proceed as above or simply rely on the relation

$$m(\angle K + \angle M) = 90°$$

Then

$$m\angle K + 46° = 90°$$

$$m\angle K = 90° - 46° = 44°$$

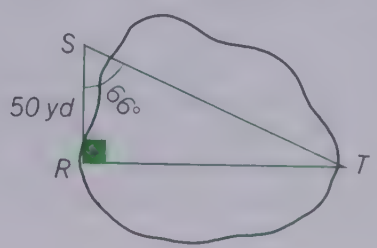
Therefore $m\angle K$ is 44°.

To solve a triangle means to calculate the measures of the unknown sides and angles.

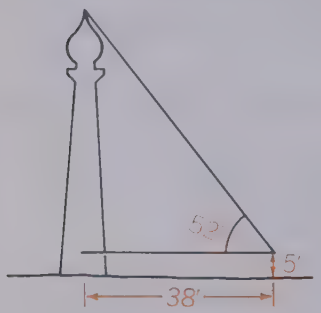
EXERCISE 6·2

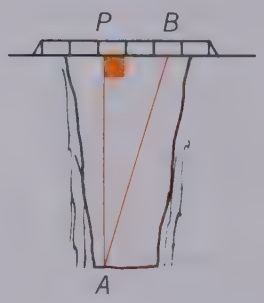
For questions 1 to 4, $\triangle ABC$ has right angle C.

1. In $\triangle ABC$, if $m\angle A = 32°$ and $c = 12$, find a.
2. In $\triangle ABC$, if $m\angle B = 20°$ and $c = 40$, find a.
3. In $\triangle ABC$, $m\angle A = 76°$ and $a = 54$. Solve the triangle.
4. In $\triangle ABC$, $m\angle B = 57°$ and $a = 21$. Solve the triangle.
5. Sue and Joan were challenged to find the distance across a small lake from point R to point T. They located a point S, 50 yards from R so that $m\angle SRT = 90°$. At S they measured angle $RST = 66°$. Find the distance RT across the lake.



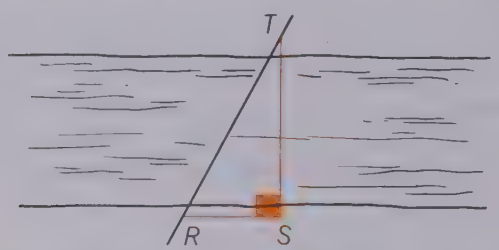
6. Bill and Dick decided to find the height of a monument in their neighbourhood park. They constructed an angle measuring device which measured vertical angles. At a point 38 feet from the middle of the monument they measured the angle to the top as 52° with the horizontal. Their instrument was 5 feet above ground level. How high is the monument?



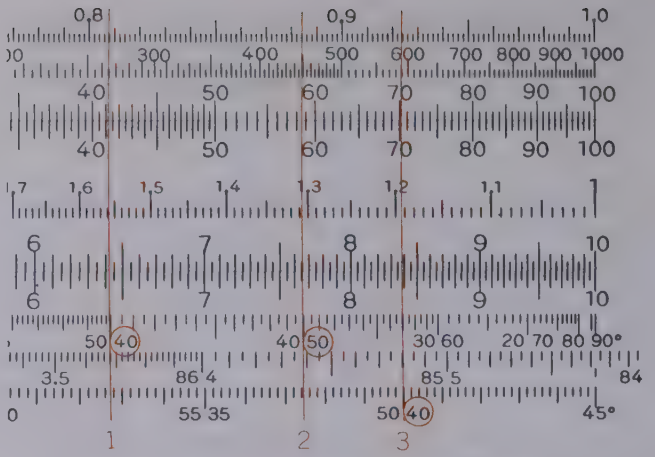


7. Two engineering students set out to determine the depth of a gorge under Bow Creek bridge. They located a reference point *A* below the bridge. With their instruments they located a point *P* on the bridge so that $\overline{AP}$ was perpendicular to the floor of the bridge. They measured off 80 feet along the floor of the bridge from *P* to *B*. They measured $\angle PBA$ to be 78°. From this information calculate (to the nearest foot) the depth of the gorge.

8. A power line survey crew had to find the span across a river from point *R* to point *T*. Their survey diagram showed $\angle TRS$ to have a measure of 68°, and $\overline{RS}$ measured 120 feet. What was the length of the span from *R* to *T*?



9. Your slide rule gives approximate values of the trigonometric functions. For sine angles use the S scale; for the cosine angles use the numerals on the S scale; and for the tangent angles use the T scale. The values for all three are found on the C or D scales.



Note that position 1 of the hairline shows
sin 40° = 0.643 and cos 50° = 0.643.

Do you notice that 40 + 50 = 90?

Also position 2 of the hairline shows
sin 50° = 0.766 and cos 40° = 0.766.

Position 3 shows tan 40° = 0.84.

Use your slide rule to find the values of:

sin 20°	cos 50°	tan 25°
sin 35°	cos 25°	tan 37°
sin 62°	cos 58°	tan 28°

Check your answers by using the tables.

For tangent angles greater than 45° use the numerals on the T scale but read the values on the C1 scale. For sine and tangent values less than 6° read the angles on the ST scale and the values on the D scale. For cosine angles more than 84° convert to sine then use the sine value.

$$\cos \theta = \sin (90 - \theta)$$

Use your slide rule to find these values, then check with your tables.

sin 3°	tan 4°	sin 2°
tan 5°	cos 86°	cos 88°

10. Construct a device for measuring vertical angles (as in question 6) and one for measuring horizontal angles (as in question 5). Take them outdoors and use trigonometric function values to find heights of buildings, towers, etc. and widths of fields, streams, etc. Compare your results with others in the class. (For more details see **Math Probe 2**.)

6·3 Applications to Obtuse and Reflex Angles

Trigonometry is very useful in dealing with angles greater than 90°, such as angles generated by a rotating unit (e.g., a generator or alternator of a car).

To accommodate angles greater than 90° we must design a method of relating them to angles of right triangles. We take the radius vector *OP* and place it so that one end is at the origin of a Cartesian coordinate grid and the other end *P* on the positive *x*-axis (see Figure 6-4).

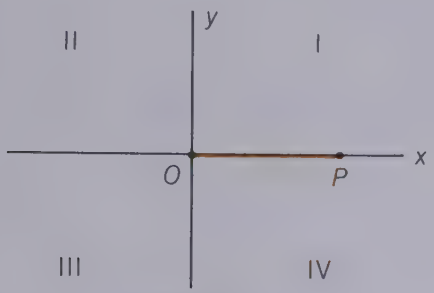


Figure 6-4

In counterclockwise direction we number the quarters of the grid as divided by the coordinate axes. Note in the diagram Figure 6-4 the labels I, II, III and IV. We call each quarter a *quadrant*.

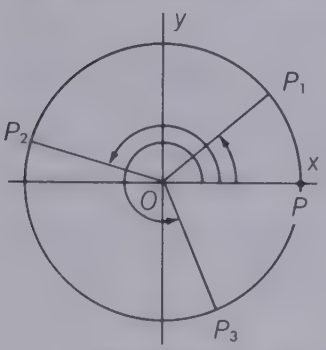


Figure 6-5

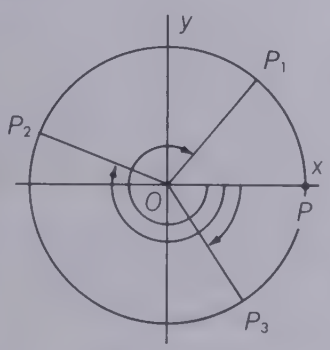


Figure 6-6

From the position *P* on the positive *x*-axis counterclockwise rotation of the radius vector *OP* about the origin *O* is considered positive. All positive rotations such as those marked P_1OP, P_2OP and P_3OP generate positive angles (refer to Figure 6-5). All rotations in a clockwise direction (as illustrated in Figure 6-6) generate negative angles. It is possible for the radius vector to make many rotations, thereby generating angles of +375°, +1000°, −720°, −1620°, etc.

Let us now consider trigonometric functions of any sized angle. In Figure 6-7 the radius vector is labelled *r* (it is always a positive magnitude). The point P_1 in the first quadrant has the coordinates (*x*,*y*).

From P_1 of the radius vector a perpendicular is drawn to the *x*-axis to form the right triangle OP_1A. The angle generated is usually labelled θ, read *theta*. We now read the functions

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \text{and}\ \tan\theta = \frac{y}{x}.$$

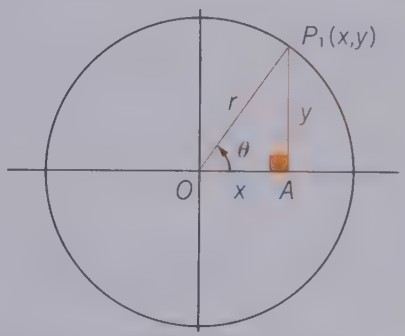


Figure 6-7

The location of a person on a Ferris Wheel can be given by the ordered pair (x,y) where y is the height from the centre of the wheel and x is the distance from the vertical line through the centre.

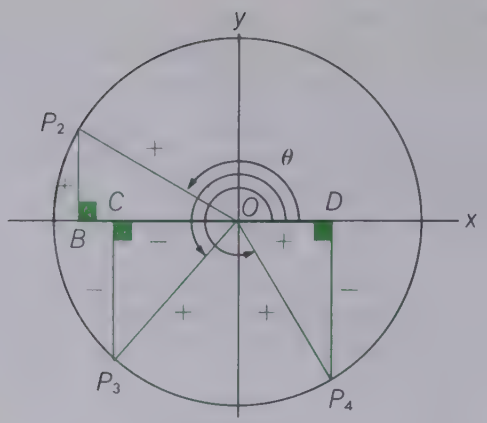


Figure 6-8

However, when the angle generated is greater than 90° (as illustrated in Figure 6-8) the functions of the angle θ are read with respect to the right triangle formed in the particular quadrant. When the radius vector r is in the 2nd quadrant, the right triangle OP_2B is formed by constructing a perpendicular from P_2 to the x-axis. (*Note!* The perpendicular from P of the radius vector OP to the x-axis forms the right triangle in each quadrant). In the 3rd and 4th quadrants the right triangles are OP_3C and OP_4D respectively. (Figure 6-8)

Let $x, y \in {}^*R$. Then the coordinates of P as the radius vector rotates about the origin O are as follows: in the 1st quadrant $P_1(x,y)$; in the 2nd quadrant $P_2(-x,y)$; in the 3rd quadrant $P_3(-x,-y)$, and in the 4th quadrant $P_4(x,-y)$.

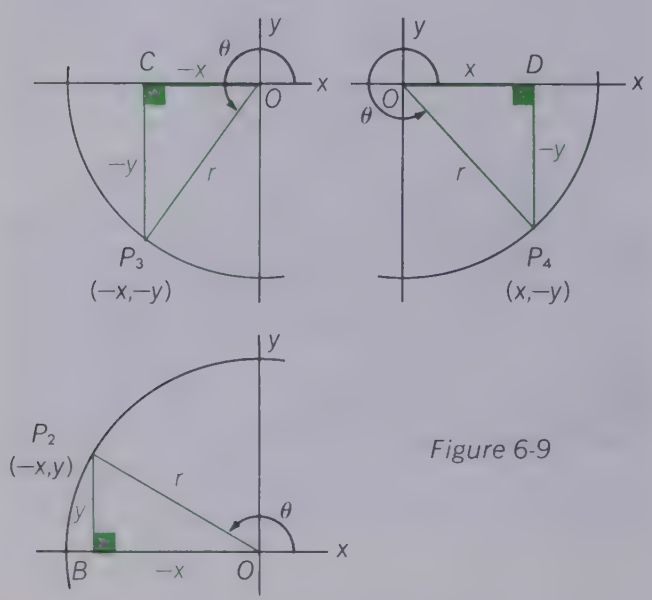


Figure 6-9

Now we can read the values of the functions of θ in each quadrant (refer to Figure 6-8).

Copy and complete the table of values.

II $90° < \theta < 180°$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = -\frac{x}{r}$ $\quad \tan\theta = -\frac{y}{x}$

III $180° < \theta < 270°$

$\sin\theta =$ ■ $\quad \cos\theta = -\frac{x}{r}$ $\quad \tan\theta =$ ■

IV $270 < \theta < 360°$

$\sin\theta =$ ■ $\quad \cos\theta =$ ■ $\quad \tan\theta =$ ■

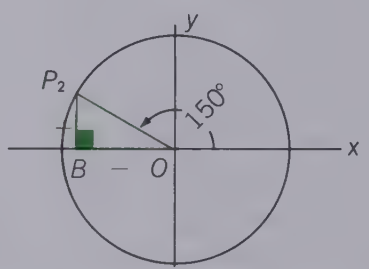
Example 1

Find the values of:

(a) sin 150° (b) tan 235° (c) cos 252°
(d) cos (−28°) (e) sin (−300°) (f) tan (−190°)
(g) tan (−1000°) (h) cos 4200°

Solution

(a)



An angle of 150° locates $\overline{OP}$ in 2nd quadrant. The triangle formed is OP_2B (as in Figures 6-8 and 6-9). The reference angle for the function is $\angle P_2OB$ the measure of which is 180° − 150° = 30°. In the 2nd quadrant the sin value is positive.

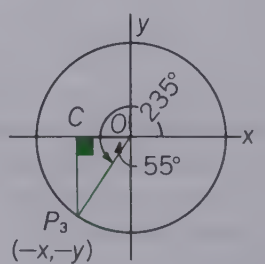
Therefore, $\sin 150° = -\sin 30°$
$= .5000$ (from tables).

(b) 235° locates $\overline{OP}$ in 3rd quadrant (see Figure 6-9). The reference angle is $\angle P_3OC$.

$m\angle P_3OC = 235° - 180°$
$= 55°$

In the 3rd quadrant the tan values are $\left(\frac{-y}{-x}\right)$ positive.

Therefore,
$\tan 235° = +\tan 55°$
$= 1.4281$



(c) 252° is a 3rd quadrant angle.

The reference angle has a measure of
$252° - 180° = 72°$.

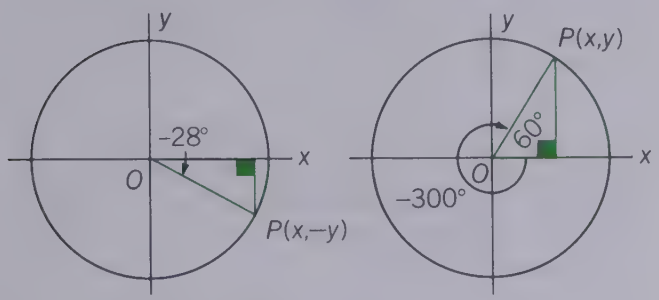
In the 3rd quadrant the cos values $\left(\frac{-x}{r}\right)$ are negative.

Therefore, $\cos 252° = -\cos 72°$
$= -.3090$

(d) $-28°$ is a 4th quadrant angle.

In the 4th quadrant the cos values $\left(\frac{x}{r}\right)$ are positive.

Therefore, $\cos(-28)° = +\cos 28°$
$= .8829$

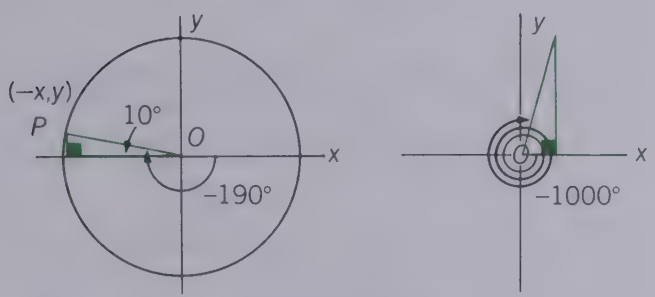


(e) $-300°$ is a 1st quadrant angle.

$\sin(-300°) = +\sin 60°$
$= .8660$

(f) $-190°$ is a 2nd quadrant angle

$\tan(-190°) = -\tan 10°$
$= -.1763$



(g) $-1000°$ is a 1st quadrant angle.

$\tan(-1000°) = +\tan 80°$
$= 5.6713$

(h) 4200° is a 3rd quadrant angle because $4200° = 11(360°) + 240°$. (Why this relation?)

$\cos 4200° = \cos 240° = -\cos 60° = -.5000$

Trigonometry is the basis of surveying.

EXERCISE 6·3

1. In what quadrant will the radius vector be when it generates an angle of:

   (a) $-12°$ (b) $100°$ (c) $200°$ (d) $-175°$

   (e) $-345°$ (f) $600°$ (g) $1810°$ (h) $-880°$?

2. What is the sign (positive or negative) of each of the following function values?

   (a) $\tan 120°$ (b) $\sin(-45°)$ (c) $\cos 80°$

   (d) $\cos(-150°)$ (e) $\sin(-150°)$ (f) $\tan 300°$

   (g) $\sin 660°$ (h) $\tan 3540°$ (i) $\cos(-700°)$

3. With the aid of trigonometric tables find the values of:

   (a) $\tan 45°$ (b) $\sin(-30°)$ (c) $\cos 120°$

   (d) $\cos(-150°)$ (e) $\tan 150°$ (f) $\tan 315°$

   (g) $\sin(-240°)$ (h) $\cos(-240°)$ (i) $\sin 400°$

   (j) $\tan(-315°)$ (k) $\cos 375°$ (l) $\tan(-12°)$

   (m) $\sin 660°$ (n) $\sin(-950°)$ (o) $\cos 1200°$

4. Copy and complete this table filling in the sign of each function for the corresponding quadrant.

	I	II	III	IV
$\sin\theta$	+			
$\cos\theta$				
$\tan\theta$				

6·4 Reciprocal Trigonometric Functions

For $\triangle ABC$ in Figure 6-10 the magnitude of a is given as 627. To solve for b, we choose

$$\tan A = \frac{a}{b}.$$

By substitution

$$\tan 49° = \frac{627}{b}$$

$$1.1504 = \frac{627}{b}$$

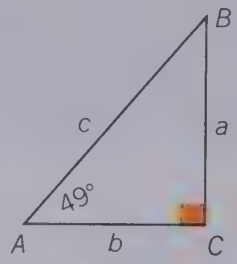


Figure 6-10

(1) $$b = \frac{627}{1.1504}$$

$$= 545 \text{ to 3 significant digits}$$

From (1) we can write $b = 627 \times \frac{1}{1.1504}$ and since $\frac{1}{1.1504} = .8693$ we can write $b = 627\ (.8693) = 545$.

It appears that multiplication by .8693 rather than division by 1.1504 would be a preferred method of computing the value of b. The obvious question comes to mind, "How does one find that number .8693 and why does it work in this case?"

It has been seen that .8693 is the reciprocal of 1.1504, i.e.,

$$\frac{1}{1.1504} = .8693 \quad \text{to 4 significant digits}$$

or $$\frac{1}{.8693} = 1.1504 \text{ to 5 significant digits}$$

Hence where division by 1.1504 was called for, multiplication (the inverse operation) by .8693 will achieve the same result.

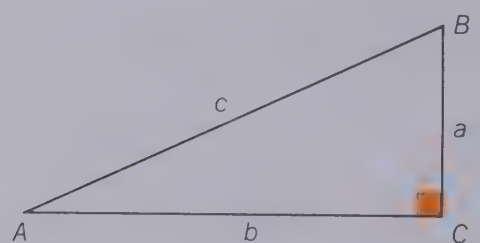


Figure 6-11

For each trigonometric function of angle A in Figure 6-11 there is a reciprocal function. Reciprocal functions give reciprocal ratios. Here they are named and their ratios are listed.

$$\sin A = \frac{a}{c} \rightarrow \csc A = \frac{c}{a}$$

$$\cos A = \frac{b}{c} \rightarrow \sec A = \frac{c}{b}$$

$$\tan A = \frac{a}{b} \rightarrow \cot A = \frac{b}{a}$$

csc is abbreviated for cosecant, sec for secant and cot for cotangent.

Hence

$$\sin A = \frac{1}{\csc A} \quad \text{or} \quad \csc \theta = \frac{1}{\sin \theta}$$

$$\cos A = \frac{1}{\sec A} \quad \text{or} \quad \sec \theta = \frac{1}{\cos \theta}$$

$$\tan A = \frac{1}{\cot A} \quad \text{or} \quad \cot \theta = \frac{1}{\tan \theta}$$

Now when you look at the Table of Values of Trigonometric Functions, you will note the csc, sec and cot values follow the first three we studied.

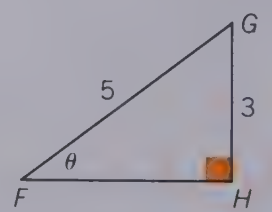
A very simple check of the Table reveals that the cos 60° = .5000 and the sec 60° = 2.0000. .5000 and 2.0000 are reciprocals of each other.

Example 1

Given $\sin \theta = \frac{3}{5}$, find the other trigonometric function values of θ.

Solution

Sketch $\triangle FGH$ with right angle at H and θ the measure of $\angle F$. Label $f = 3$ and $h = 5$ so that $\sin \theta = \frac{3}{5}$.



Pythagoras stated this relation for the Right Triangle:

$h^2 = f^2 + g^2$ *for accompanying triangle*

$5^2 = 3^2 + g^2$ *by substitution*

$g^2 = 25 - 9$

$g = 4$ *Why do we discard −4?*

Therefore, the trigonometric function values of θ are

$\sin\theta = \frac{3}{5}$ $\quad \cos\theta = \frac{4}{5}$ $\quad \tan\theta = \frac{3}{4}$

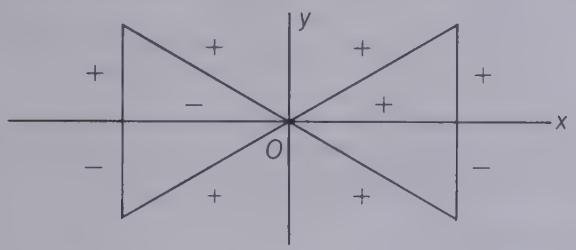
$\csc\theta = \frac{5}{3}$ $\quad \sec\theta = \frac{5}{4}$ $\quad \cot\theta = \frac{4}{3}$.

Example 2

Given $\cos\theta = \frac{12}{13}$ and $\cot\theta$ is negative. Find $\sin\theta$, $\tan\theta$, $\csc\theta$.

Solution

From a sketch of right triangles in the four quadrants we observe that θ can be a 1st quadrant or 4th quadrant angle for $\cos\theta$ to be positive. $\cot\theta$ is positive in 1st quadrant and negative in 4th quadrant. Hence θ is a 4th quadrant angle.



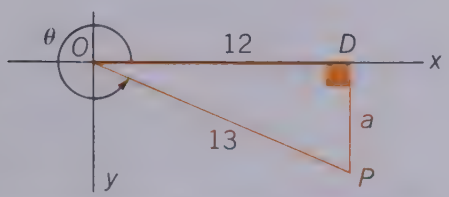
In the rt$\triangle OPD$,

$13^2 = a^2 + 12^2$

$169 = a^2 + 144$

$a^2 = 25$

$a = -5$



Because a is an ordinate in 4th quadrant, it has a negative value.

Hence, $\sin\theta = \frac{-5}{13}$, $\tan\theta = \frac{-5}{12}$ and $\csc\theta = -\frac{13}{5}$.

What are the values for $\sec\theta$ and $\cot\theta$?

EXERCISE 6·4

1. Sketch a right triangle in each of the quadrants (as in Section 6-4, Example 2). Label the sides as + or − for each particular quadrant. Recall the radius vector is +.
   (a) In which quadrant are all functions positive?
   (b) In which quadrants are $\tan\theta$ functions negative?

If a curling rock lies at 2 o'clock on the outer ring, how much farther should it have gone to be a "bull's eye"? What information will you need?

   (c) What functions are positive in the 3rd quadrant?
   (d) In which quadrant are $\cos\theta$ and $\cot\theta$ negative?
   (e) In which quadrants is $\sec\theta$ negative?
   (f) How many functions are positive in the 2nd quadrant? Name them.
   (g) Which functions are positive in the 3rd quadrant? in the 4th quadrant?

2. Make a table and record the signs of each of the 6 functions in each of the 4 quadrants.

3. Write a jingle or a mnemonic aid to assist you to remember which functions are positive in each of the four quadrants.

4. If θ is a 3rd quadrant angle and $\cos\theta = \frac{3}{5}$, find $\sin\theta$ and $\tan\theta$.

5. If θ is a 2nd quadrant angle and $\tan\theta = \frac{12}{5}$, find $\csc\theta$ and $\sec\theta$.

6. If $\cot\theta$ is $\frac{1}{\sqrt{3}}$ and $\sin\theta$ is negative, find $\cos\theta$ and $\csc\theta$.

7. If $\csc\theta$ is $\frac{-\sqrt{2}}{1}$ (or $-\sqrt{2}$) and $\cot\theta$ is negative, find $\cos\theta$ and $\tan\theta$.

8. If $\cos\theta = -\frac{4}{5}$ and $\tan\theta$ is positive find the value of $\frac{2\tan\theta}{1-\tan^2\theta}$. $[\tan^2\theta = (\tan\theta)^2]$

9. If $\tan\theta = \sqrt{3}$ and $\cos\theta$ is negative, find the value of $\sin^2\theta + \cos^2\theta$.

Project

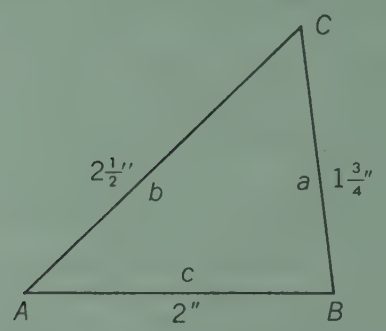
Project 1

Purpose

To illustrate the Law of Sines.

Method

1. Draw with extreme care this triangle.

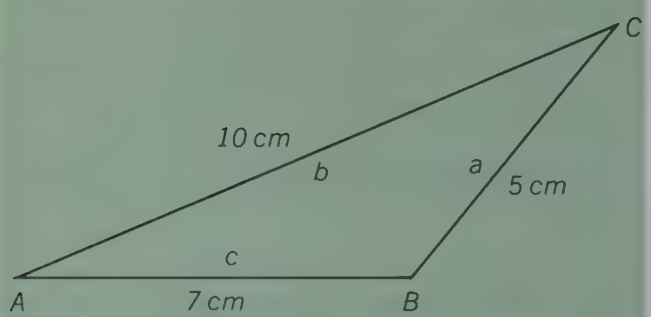


2. Measure carefully each angle.

3. Calculate to one decimal place each ratio; use tables for the trigonometric functions – round each to two significant figures.

$$\frac{\sin A}{a} \quad \frac{\sin B}{b} \quad \frac{\sin C}{c}$$

4. Repeat steps 1 to 3 for this triangle:



Questions

1. What observation do you make of the three ratios for each triangle? Discuss this with your classmates.

2. If you feel it necessary, confirm your conclusion by testing on a triangle of your choice.

6·5 Law of Sines

Trigonometric functions have proven of value in solving triangles other than right triangles. One relation of such value is called the Law of Sines.

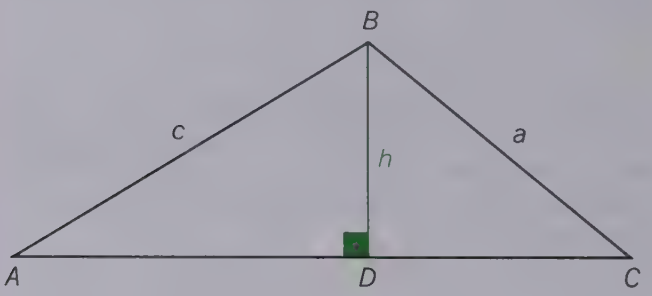


Figure 6-12

For any triangle ABC (Figure 6-12), from B a perpendicular to $\overline{AC}$ at D is constructed.

In right $\triangle ABD$

(1) $\sin A = \frac{h}{c}$, so that $h = c \sin A$

In right $\triangle CBD$

(2) $\sin C = \frac{h}{a}$, so that $h = a \sin C$

From (1) and (2)

$c \sin A = a \sin C$ *Why?*

This may be written in the form

$$\boxed{\frac{\sin A}{a} = \frac{\sin C}{c}}$$

By a similar treatment it can be shown that

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b}}$$

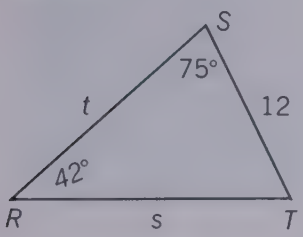
Hence, for any $\triangle ABC$

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

Example 1

For $\triangle RST$, if $m\angle R = 42°$, $m\angle S = 75°$ and $r = 12$, find $m\angle T$ and sides s and t.

Solution



$$m\angle T = 180° - (42° + 75°)$$
$$= 180° - 117°$$
$$= 63°$$

To find side s

$$\frac{\sin R}{r} = \frac{\sin S}{s}$$

$$\frac{\sin 42°}{12} = \frac{\sin 75°}{s}$$

$$s = \frac{12 \sin 75°}{\sin 42°}$$

$$= \frac{12\,(.9659)}{.6691}$$

$$= 17.3$$

Side s is 17.3 units.

To find side t

$$\frac{\sin R}{r} = \frac{\sin T}{t}$$

$$\frac{\sin 42°}{12} = \frac{\sin 63°}{t}$$

$$t = \frac{12 \sin 63°}{\sin 42°}$$

$$= \frac{12\,(.8910)}{.6691}$$

$$= 16.0$$

Side t is 16.0 units.

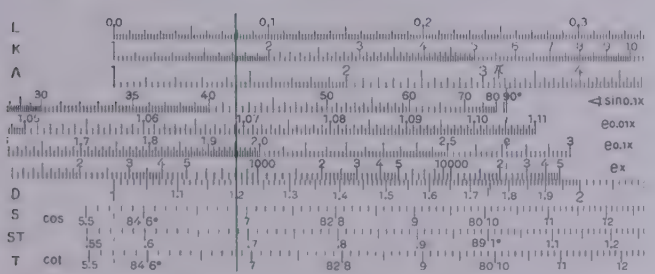
The problems involving the Law of Sines can be easily solved using the slide rule. To solve the problem in Example 1 we can write

$$\frac{\sin R}{r} = \frac{\sin S}{s} = \frac{\sin T}{t}$$

or

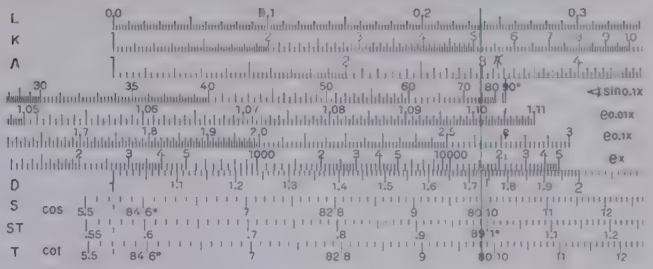
$$\frac{\sin 42°}{12} = \frac{\sin 75°}{s} = \frac{\sin 63°}{t}.$$

Now using the slide rule, set the hairline over 12 on the D scale. Then slip the slide until the sin 42° is under the hairline.



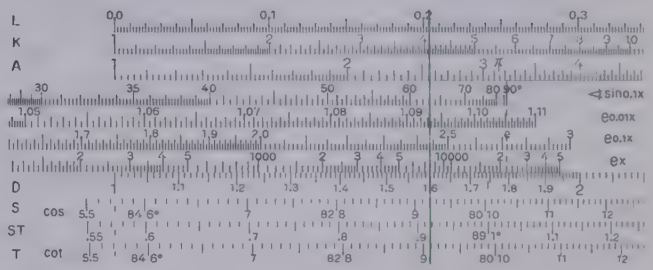
With this setting of the slide rule we can read the answer for both s and t by simply making two adjustments of the hairline.

Move the hairline over sin 75° to find s on the D scale.



The answer is 17.3

Now move the hairline over sin 63°. Read the answer on the D scale.

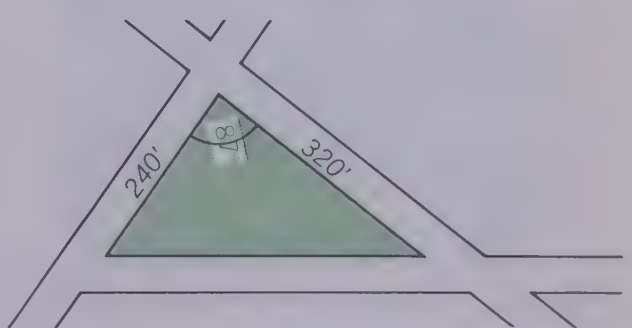


The answer is 16.

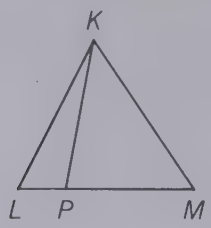
EXERCISE 6·5

1. Solve each of the following triangles for the remaining parts.
   (a) In $\triangle ABC$, $a = 75$, $m\angle B = 60°$, $b = 100$.
   (b) In $\triangle PQR$, $m\angle Q = 49°$, $m\angle R = 77°$, $r = 400$.
   (c) In $\triangle XYZ$, $m\angle X = 82°$, $m\angle Y = 33°$, $z = 300$.

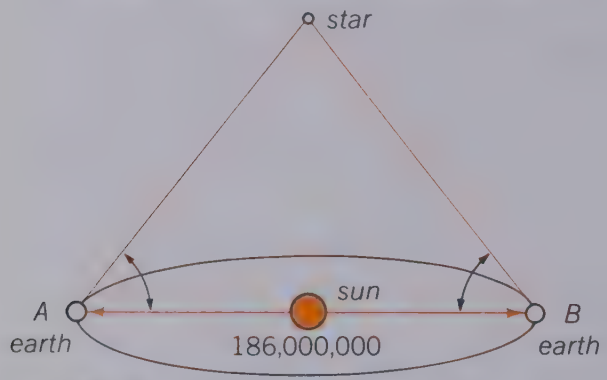
2. A triangular lot as illustrated in this diagram has the indicated measurements. Find the measures of the remaining two angles and sides.



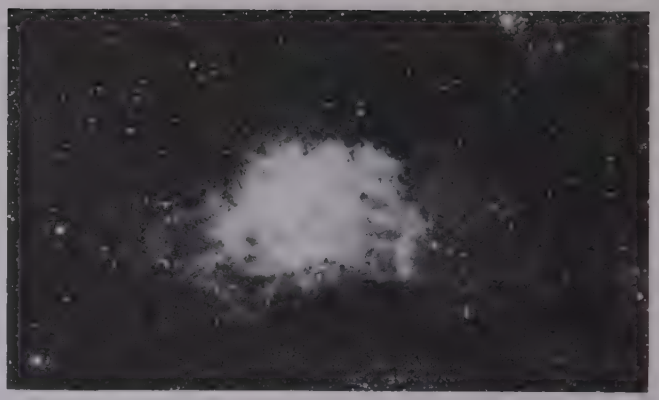
3. Equilateral $\triangle KLM$ has sides each measuring 1000 feet. If P is on $\overline{LM}$ so that $m\angle KPM = 71°48'$, find the measure of $\overline{LP}$.



4. Triangulation is a method used in calculating the distance to stars. The diameter of the earth's orbit (roughly) is 186,000,000 miles. This is used as the base of the triangle. The two base angles are obtained by observation. If $m\angle A = 86°$ and $m\angle B = 88°$ find the distance from B to the star.



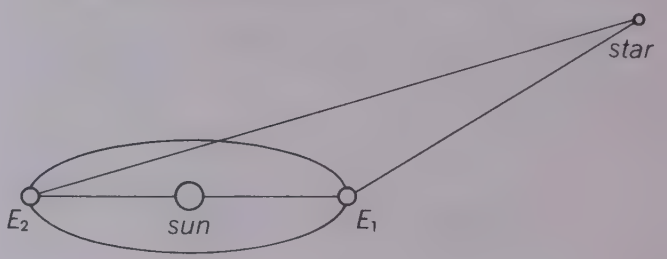
Technician operates radio telescope used to calculate position and distance of the stars.



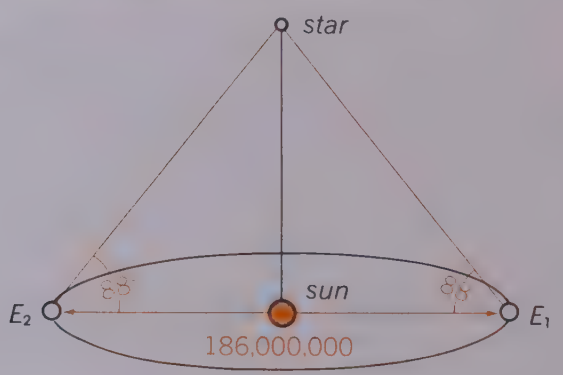
5. In a parallelogram two adjacent sides measure 130 feet and 200 feet. The included angle measures 74°. Find the lengths of the two diagonals of the parallelogram.

6. By observation, the two base angles of the triangle when "shooting" a second star are $m\angle A = 87°$ and $m\angle B = 89°$. What is the difference in the two distances from the earth to the star?

7. Astronomers "shoot" a third star. Angle E_2 was found to be 13°, angle E_1 was 166°. Calculate the distance E_2 is from the star.



8. After shooting a heavenly body astronomers recorded the information on this diagram. Calculate the distance from the sun to the body.



Using the Library

1. Continue your mural by adding these events and names:

   Ptolemy
   Marcus Aurelius – sends embassy to China
   Diophantus – equations
   Pappus
   Council of Nicaea
   Founding of Constantinople
   St. Jerome writes the Vulgate
   Sacking of Rome by the Goths
   Hypatia – first woman of mathematics
   Eutocius – writer of first standard text
   Jerusalem taken by the Arabs
   Brahmagupta – a no. < 10
   Muslims invade Spain
   Charlemagne's reign
   Caliph Hārūn al Rashīd – Tales of the Arabian Nights
   Muhammad ibn Mūsā al Khwarizmi – first person to use the word algebra
   Canute

2. Little is known specifically of the details of Hypatia's death though accounts in general are well known. Research the life of Hypatia and the times in which she lived. Then write a fictionalized account for radio or TV of her death reflecting the superstitions of that day.

3. D. E. Smith in his book *History of Mathematics*, Volume I, page 248 says "The five centuries extending from 500 to 1000 saw the general trend of mathematics to the West rather than in the opposite direction. Europe was intellectually dormant drugged with a new narcotic, while most of the East was, as always superstitious but inquisitive."

   Explain what is meant by the underlined portion. *Hint!* What is the common name for this period of time in Europe?

4. Mathematicians often become embroiled in the political and religious problems of the days in which they lived. Describe the involvement of Boethius and his ultimate end.

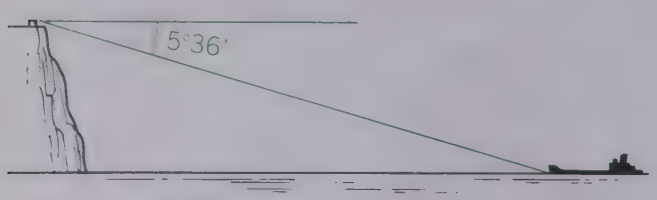
5. The Indian Aryabhata is said to have given this method of calculating the next triangle in the chart here.

   Compare this value calculated to four decimal places with the value of π as known today.

6. Comment on the statement "The Church viewed with suspicion all attempts at mathematical knowledge among the Dark Ages."

REVIEW EXERCISE 6

1.



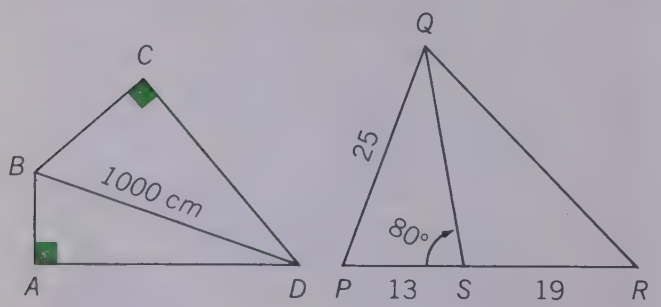
From an observation post on top of a cliff a ship was sighted and the angle of depression was measured to be 5° 36′ to the waterline of the ship. If the observation post was 528 feet above sea level, how many miles was it from the post to the ship (along the line of sight)?

2. Find the values of:
   (a) cos 120° (b) tan 315° (c) csc (−210°)
   (d) cot 955° (e) sin (−390°) (f) sec 7000°

3. Given $\tan\theta = \frac{12}{5}$, $0° < m\angle\theta < 90°$, find the other trigonometric function values of θ.

4. Given $\csc\theta = -\frac{5}{3}$ and $\tan\theta$ is positive. Find the other trigonometric function values of θ. Find the value of $\dfrac{1-\sin\theta}{\cos\theta}$.

5. In quadrilateral $ABCD$, $\angle A$ and $\angle C$ are right angles. Diagonal BD is 1000 centimetres long. The measure of $\angle CBD = 65°$ and $\angle ABD = 60°$. Find the perimeter of the quadrilateral.



6. In $\triangle PQR$, the measure of $\angle PSQ$ is 80°. The other measures are as follows:
   $m(\overline{PS}) = 13$ $m(\overline{PQ}) = 25$ $m(\overline{RS}) = 19$.
   Find the perimeter of $\triangle PQR$.

Your Job and Mathematics

Riverdale Farm Co. Ltd.
Cochrane, Alberta.

Any young person who is going to farm, either for himself or for somebody else must be versed in many areas – mathematics being one – for farming is becoming a very competitive industry.

1. A farmer wishes to estimate the cost of his seeding operation. He has 2260 acres to put into wheat. He wishes to sow 1.5 bushels per acre and to apply 6.5 pounds of fertilizer per acre. The cost of the treated wheat seed is $4.50 per bushel. The fertilizer is $3.75 per 100 pounds. Calculate the farmer's outlay of cash just for this portion of his planting operation. In addition to this the cost of running his machinery and of hired help comes to 62¢ per acre. Calculate the farmer's cost of planting according to these statistics.

2. A farmer has this offer made to him:

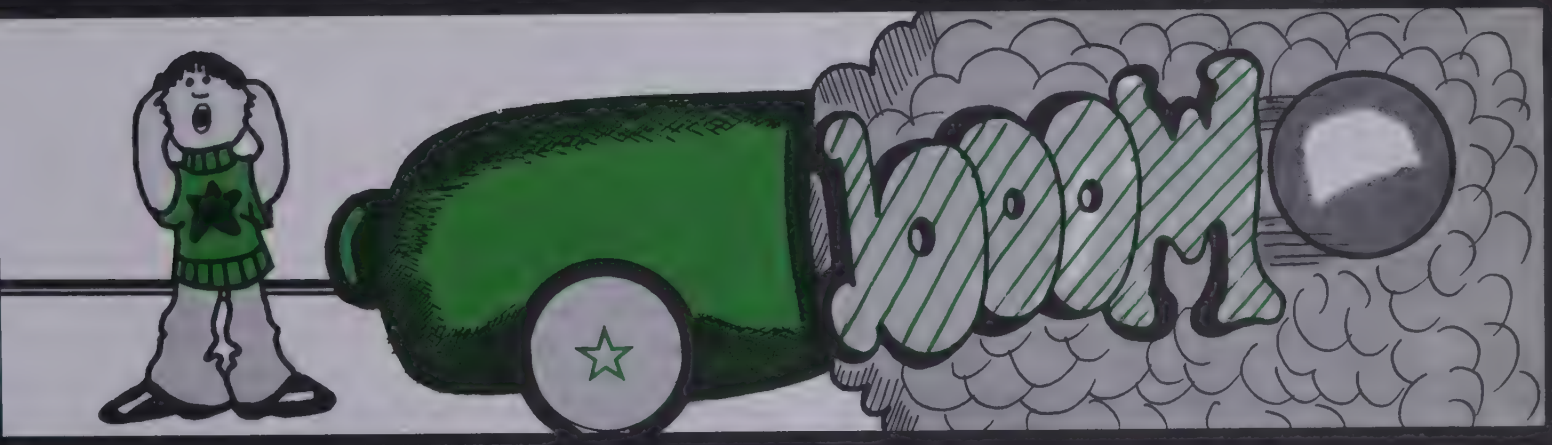
   Company A will provide him with 40 young steers each weighing on the average of 450 pounds. The farmer is to feed his own oats which on the market sell at 75¢ per bushel net (if the "quota" opens so he can sell it). It cost $3\frac{1}{2}$¢ per bushel to treat the oats and to hammer it into feed. In addition roughage (straw or hay) costs 30¢ a bale each weighing about 50 pounds. A steer needs about 3 bales per week. The company will pay the farmer 18¢ for each pound gain shown by the steers. The farmer will have to spend $300 to fix up his corrals and arrange for feeding sheds. Hopefully the steers will weigh 900 lbs each in six months.

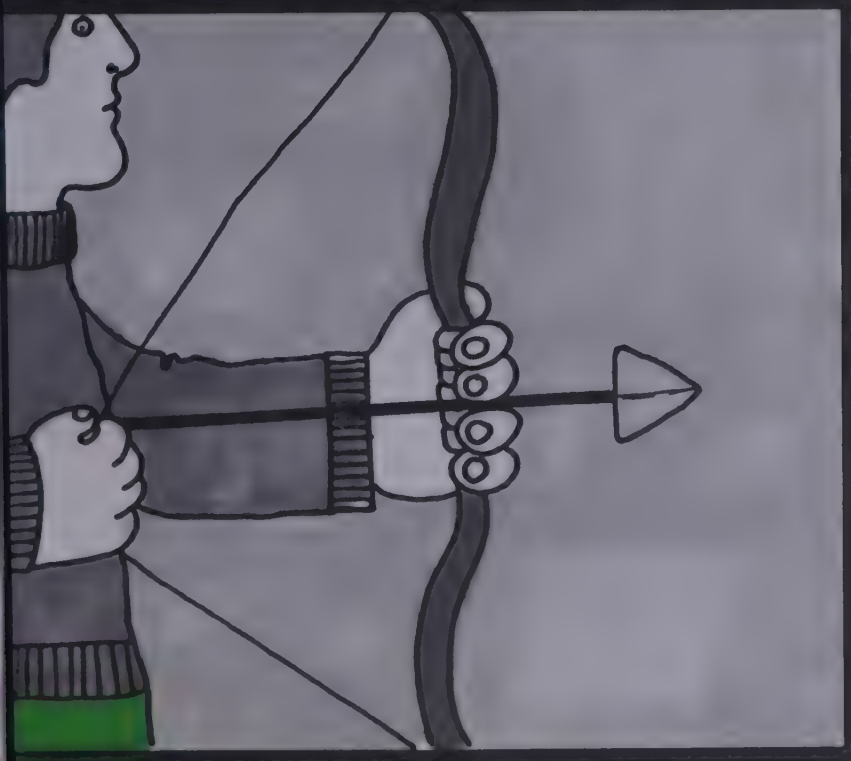
   (a) Should the farmer accept the offer?
   (b) What additional information do you need?
   (c) Contact your district agriculturist or department of agriculture to gain the information to answer this question.
   (d) If the answer is "no" to (a) what price should the farmer get per pound gain on each steer to make the deal profitable?

Remember! If you were the farmer you would stand to lose a lot of money or to make a lot. So calculate everything carefully.

Chapter Seven

VECTORS IN USE





7·1 Vectors in Two Dimensions

In this chapter we shall review briefly vectors in two dimensions. If problems are encountered, you should refer to **Math Probe 3**, Chapter 8.

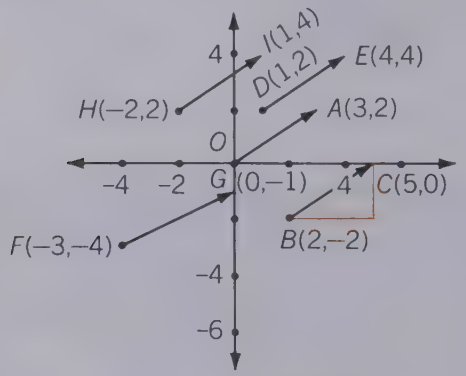


Figure 7-1

Each of the vectors in Figure 7-1 has a horizontal displacement and a vertical displacement. Copy and complete this chart for the vectors in Figure 7-1.

Vector	Horizontal displacement	Vertical displacement
$\vec{OA}$ $\vec{BC}$ $\vec{DE}$ $\vec{HI}$ $\vec{FG}$	5 – 2 = 3 units	0 – (–2) = 2 units

What do you notice about the displacement of the vectors?

The *magnitude* and the *direction* are the same for all the vectors. Such vectors are *equivalent*. We may represent all equivalent vectors by the vector in standard position, $\vec{OA}$. The algebraic representation of this standard vector is (3,2) – the ordered pair of its endpoint. If a vector is not in standard position we identify it by naming the ordered pairs of its initial point (first), then its endpoint (second). Hence $\vec{BC}$ is named [(2,–2), (5,0)] where (5,0) is its endpoint and (2,–2) is its initial point.

Addition of Vectors

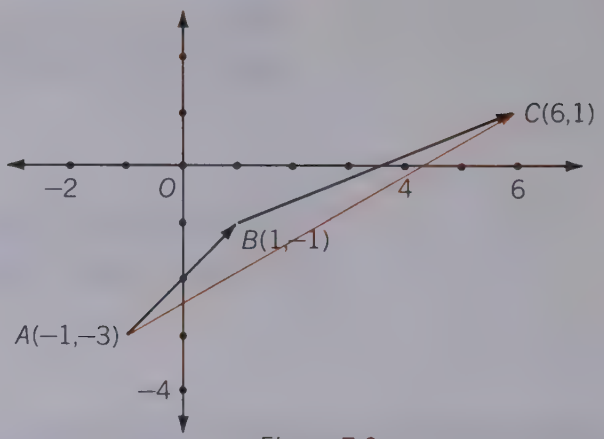


Figure 7-2

We wish to add $\vec{AB}$ and $\vec{BC}$ in Figure 7-2. We shall do it in two ways.

The first is from basic principles.

$$\vec{AB} + \vec{BC} = [(-1,-3), (1,-1)] + [(1,-1), (6,1)]$$
$$= [(-1,-3), (6,1)]$$

This means $\vec{AB} + \vec{BC} = \vec{AC}$. To interpret this, we say $\vec{AC}$ is equivalent to the sum $(\vec{AB} + \vec{BC})$, or the sum of the two vectors $(\vec{AB} + \vec{BC})$ is that one vector that has the same displacement as the sums. We notice that the sum vector has a horizontal displacement of 7 and a vertical displacement of 5. Hence, it may be described by [(0,0), (7,5)] or just (7,5).

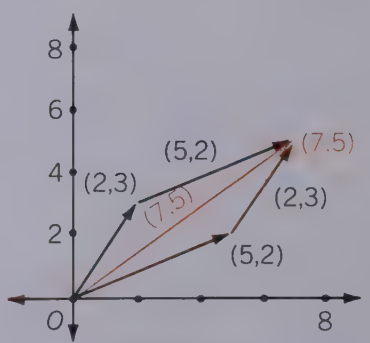


Figure 7-3

The second method involves the names (algebraic names) of the equivalent vectors in standard position. Note $\vec{AB}$ has a horizontal displacement of 2 and a vertical displacement of 3. Hence its standard name (algebraic) is (2,3). The algebraic name for $\vec{BC}$ is (5,2).

The sum of these two vectors is that vector in standard position with endpoint (7,5). Hence,

$$(2,3) + (5,2) = (7,5)$$

Algebraically, we add two vectors in standard position this way:

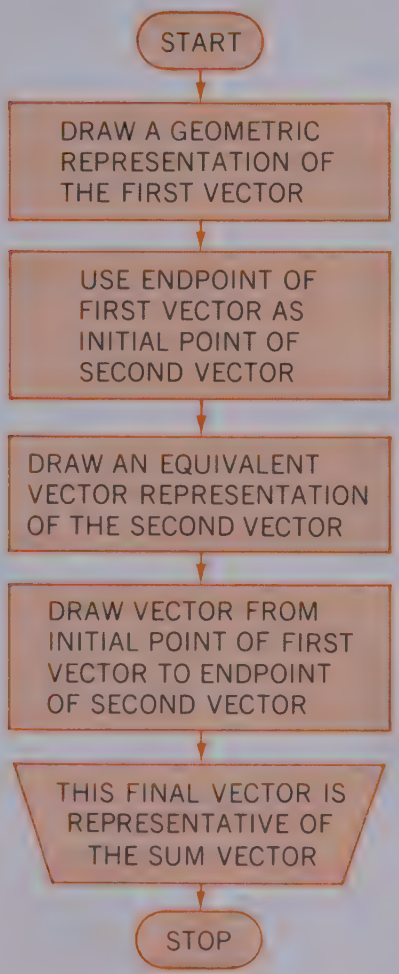
$$(a,b) + (c,d) = (a + c, b + d).$$

Do questions 1 and 2 in Exercise 7·1.

Two geometric methods of adding vectors are illustrated here.

The Triangle Law of Vectors

To add two vectors:


START
DRAW A GEOMETRIC REPRESENTATION OF THE FIRST VECTOR
USE ENDPOINT OF FIRST VECTOR AS INITIAL POINT OF SECOND VECTOR
DRAW AN EQUIVALENT VECTOR REPRESENTATION OF THE SECOND VECTOR
DRAW VECTOR FROM INITIAL POINT OF FIRST VECTOR TO ENDPOINT OF SECOND VECTOR
THIS FINAL VECTOR IS REPRESENTATIVE OF THE SUM VECTOR
STOP

Vectors may be used to analyse this physical system.

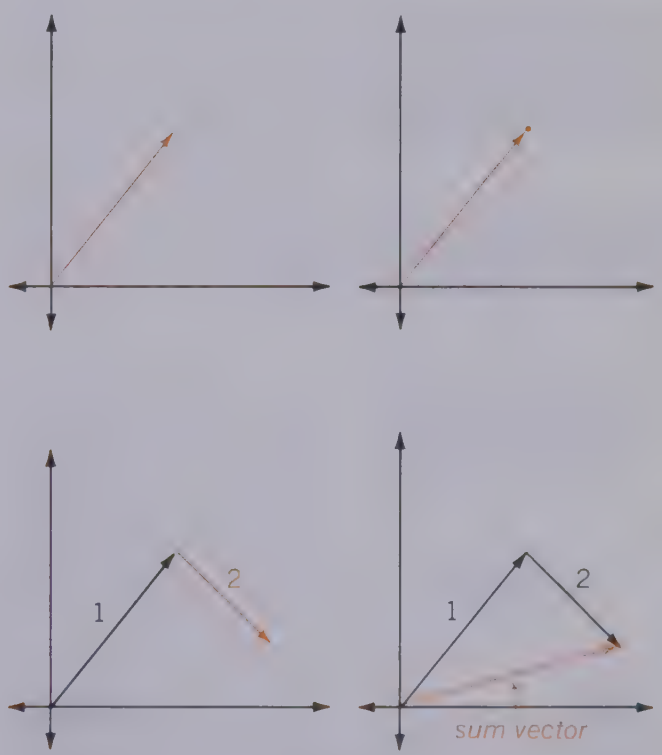


Figure 7-4

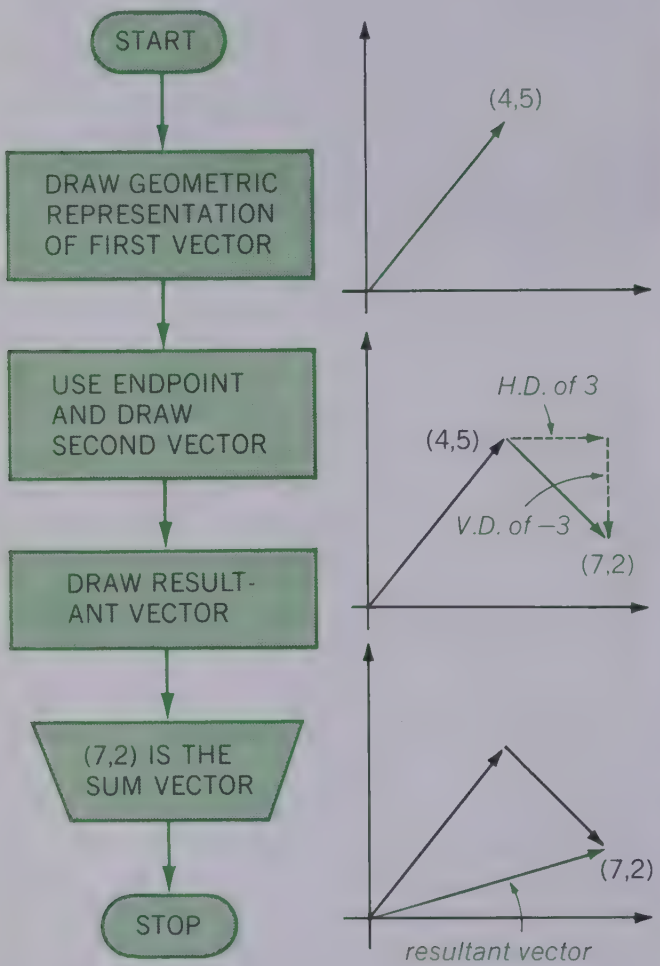
This final vector is called the *resultant* of the other two vectors.

Example 1

Add geometrically the vectors (4,5) and (3,–3).

Solution

Recall vector (4,5) means a horizontal displacement of 4 and a vertical displacement of 5 while (3,–3) means a horizontal displacement of 3 and a vertical displacement of –3.



Do question 3 of Exercise 7·1.

The Parallelogram Law of Vectors

To add two coplanar vectors:

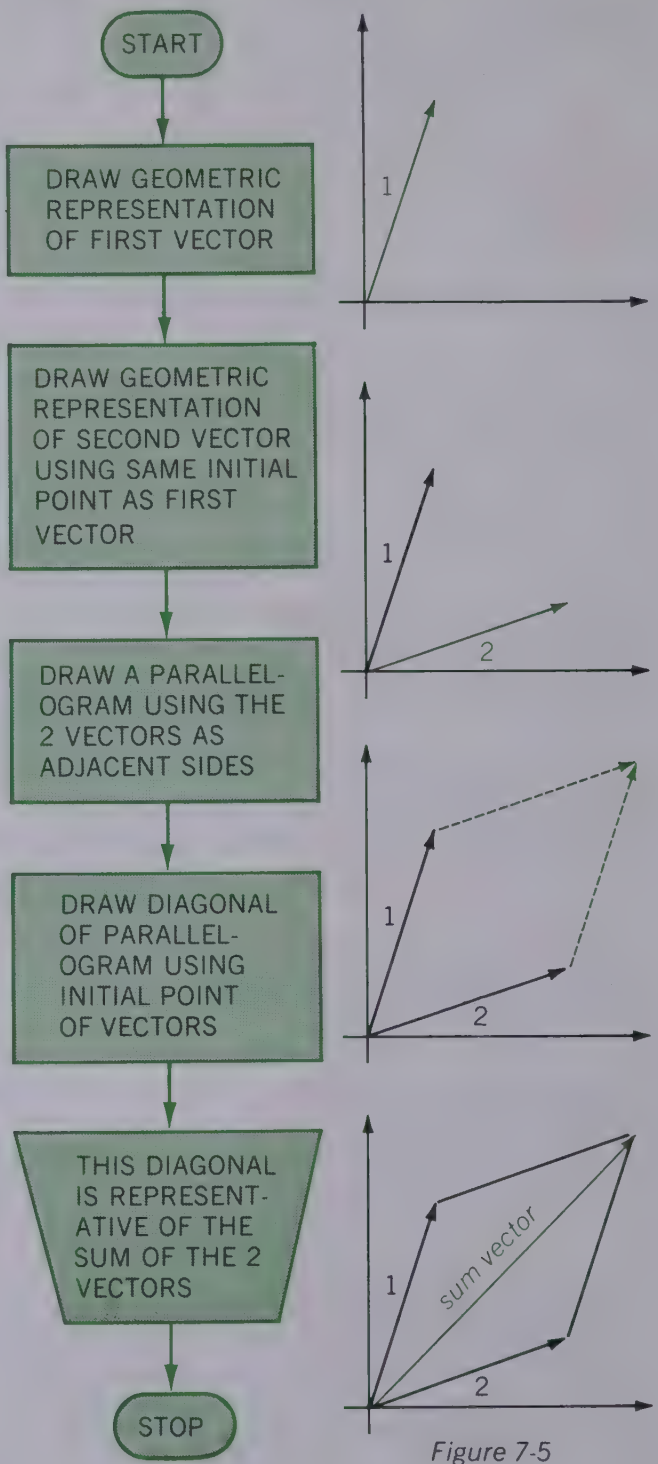

START
DRAW GEOMETRIC REPRESENTATION OF FIRST VECTOR
DRAW GEOMETRIC REPRESENTATION OF SECOND VECTOR USING SAME INITIAL POINT AS FIRST VECTOR
DRAW A PARALLELOGRAM USING THE 2 VECTORS AS ADJACENT SIDES
DRAW DIAGONAL OF PARALLELOGRAM USING INITIAL POINT OF VECTORS
THIS DIAGONAL IS REPRESENTATIVE OF THE SUM OF THE 2 VECTORS
STOP
1
1
2
1
2
1
sum vector
2

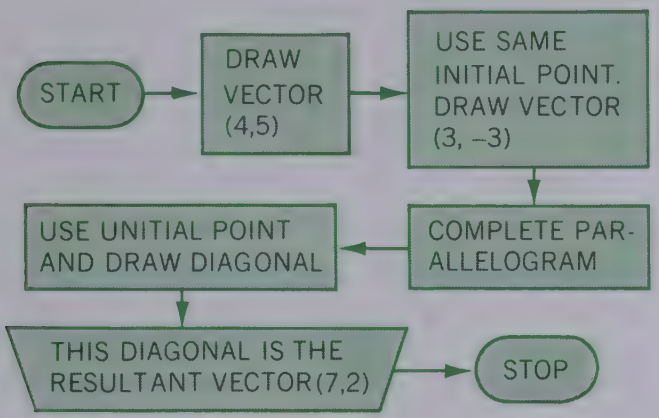
Figure 7-5

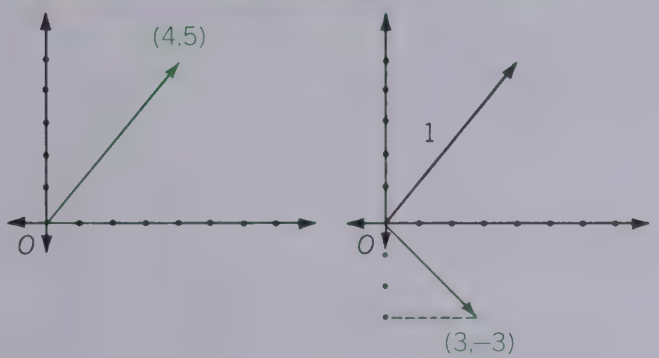
This vector which is the sum of the two other vectors is called the *resultant vector.*

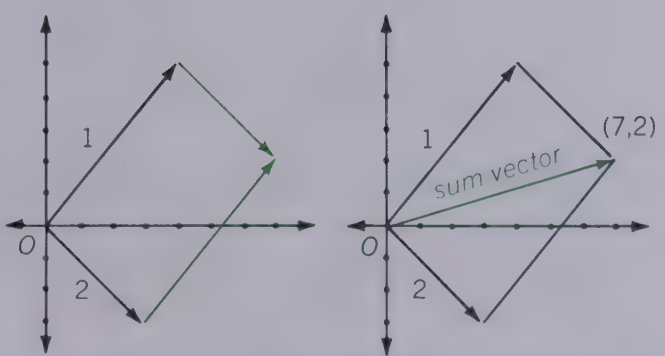
Example 2

Add geometrically the vectors (4,5) and (3,–3).

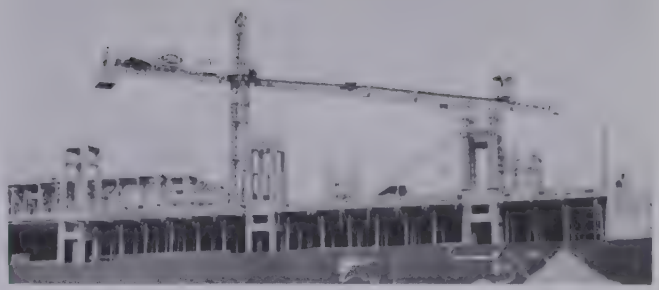
Solution







Do question 4 in Exercise 7·1.



A knowledge of vectors is essential to the engineer in order to study the stresses and strains in any large construction project.

Subtraction of Vectors

The zero vector (the additive identity) is (0,0) since $(a,b) + (0,0) = (a,b)$ where (a,b) is a vector. The additive inverse of the vector (a,b) is that vector (x,y) such that

$$(a,b) + (x,y) = (0,0).$$

Hence we know

$$a + x = 0$$

and

$$b + y = 0$$

and

$$x = -a \text{ and } y = -b.$$

Therefore, the additive inverse of (a,b) is $(-a,-b)$.

Check

$$(a,b) + (-a,-b) = (a-a,\ b-b)$$
$$= (0,0)$$

Subtraction can be defined as

$$(a,b) - (c,d) = (a,b) + (-c,-d)$$
$$= (a-c,\ b-d).$$

Example 3

Subtract (3,–5) from (7,6).

Solution

$$(7,6) - (3,-5) = (7,6) + (-3,5)$$
$$= (7-3,\ 6+5)$$
$$= (4,11)$$

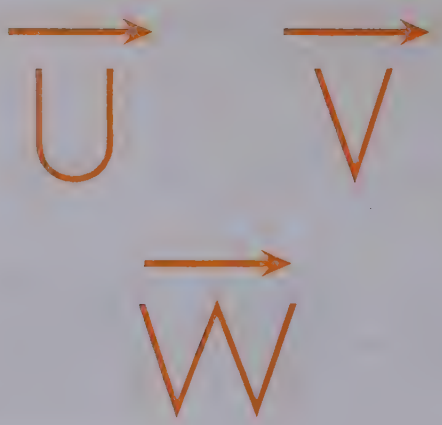
Do question 5 in Exercise 7·1.

Scalar Multiplication of Vectors

If we add (4,5) + (4,5) + (4,5) we get (12,15). Another way to do this is by *scalar multiplication.*

$$3(4,5) = (3 \times 4, 3 \times 5)$$
$$= (12,15)$$

In this case 3 is considered a scalar as it has no direction and (4,5) is a vector so we are multiplying a vector by a scalar.



Example 4

Show geometrically and algebraically the sum $2[(1,4) + (4,1)]$.

Solution

Algebraically:

Method 1

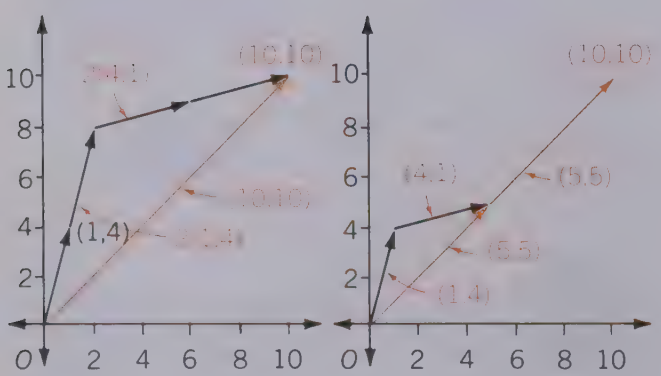
$$2[(1,4) + (4,1)] = 2(1,4) + 2(4,1)$$
Distributive Property
$$= (2,8) + (8,2)$$
$$= (10,10)$$

Method 2

$$2[(1,4) + (4,1)] = 2(5,5)$$
$$= (10,10)$$

Geometrically:

Method 1 Method 2



To simplify the writing of vectors we sometimes represent these by small case letters with an arrow above them. Symbols such as $\vec{u}, \vec{v}, \vec{w}$ are often used. The zero vector, the additive identity, is represented by $\vec{0}$ or (0,0).

If $\vec{v} = (2,1), \vec{u} = (3,6)$,
then $\vec{v} + \vec{u} = (2,1) + (3,6)$
$= (5,7)$

EXERCISE 7·1

1. State the algebraic vector represented by $\vec{AB}$ if
   (a) $A = (3,1), B = (7,4)$
   (b) $A = (5,-3), B = (2,-1)$
   (c) $A = (-6,-2), B = (-3,-2)$
   (d) $A = (5,-8), B = (-2,-5)$
   (e) $A = (7,-6), B = (-9,-3)$

2. State the endpoint of the vector (3,–7) if the initial point is
   (a) (5,2) (b) (7,9) (c) (–3,4)
   (d) (–5,–3) (e) (17,–27) (f) (–22,41)

3. Use the Triangle Law of Vectors to add each:
   (a) (4,1) + (2,6) (b) (5,–2) + (1,–4)
   (c) (–4,1) + (5,3) (d) (8,12) + (–2,–8)
   (e) (17,8) + (–17,–8) (f) (7,–7) + (–7,7)

4. Use the Parallelogram Law of Vectors to add each pair of vectors in question 3.

5. Perform the following subtractions:

   (a) $(5,6) - (3,5)$ (b) $(8,9) - (-5,6)$

   (c) $(15,6) - (-7,-9)$ (d) $(-6,9) - (-3,6)$

   (e) $(-7,-9) - (-3,-4)$ (f) $(d,f) - (m,n)$.

6. Show geometrical representation of question 5(a) and (b).

7. Represent each as a single vector (i) geometrically (ii) algebraically.

   (a) $(2,-3) + 3(4,5)$ (b) $2[(3,2) - (4,8)]$

   (c) $3[(2,-7) - (-7,-1)]$ (d) $3(2,-7) - 5(-7,-1)$

8. Represent as a single vector:

   (a) $h(a,b) + m(a,b)$ (b) $n(c,d) + n(c,d)$

   (c) $m(e,f) + m(a,b)$ (d) $s(g,h) + s(b,d)$

9. By letting $\vec{v}$, $\vec{u}$ and $\vec{w}$ represent specific vectors illustrate each of the following properties of vectors. Name each property.

   Hint! Let $\vec{v} = (3,1)$, $\vec{u} = (1,4)$, $\vec{w} = (2,2)$

   (a) $\vec{v} + \vec{u} = \vec{u} + \vec{v}$

   (b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

   (c) $\vec{w} + \vec{0} = \vec{0} + \vec{w} = \vec{w}$

   (d) $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$
   (Let k be a scalar, such as 2 or 3.)

   (e) $kl(\vec{w}) = k(l\vec{w})$, k,l are scalars

   (f) $(k + l)\vec{w} = k\vec{w} + l\vec{w}$, k,l are scalars

10. If $k = 3$, $l = -5$, $\vec{v} = (2,3)$, $\vec{u} = (4,-1)$, $\vec{w} = (-5,6)$ calculate each:

    (a) $-k\vec{v} - l\vec{u}$ (b) $k(\vec{u} + \vec{v}) + k(\vec{u} - \vec{v})$

    (c) $k(\vec{v} - \vec{u} + \vec{w})$ (d) $l(\vec{v} - \vec{w}) - k(2\vec{u} - \vec{v})$

11. Given the vector $\vec{v} = (-3,6)$, find

    (a) $2\vec{v}$ (b) $\frac{1}{3}\vec{v}$

    (c) $-4\vec{v}$ (d) $3\vec{v} + 2\vec{v}$.
    (Do this one two ways.)

12. Find the vectors represented in each part of question 1 algebraically.

13. Find the following vector sums, differences or scalar products geometrically.

    (a) $(1,1) + (1,3)$ (b) $(0,0) - (3,5)$

    (c) $(-1,0) + (2,5)$ (d) $4(-1,1)$

    (e) $-2(-1,-3)$ (f) $(3,4) - (4,6)$

    (g) $(1,1) - (4,4)$ (h) $(-1,-3) - (2,-3)$

    (i) $\frac{1}{2}(3,0) + (5,-4)$ (j) $\frac{1}{5}(10,4) - \frac{1}{5}(20,6)$

14. Perform the following problems by vector diagrams drawn to scale.

    (a) Two coplanar forces act on a body: a force of 60 pounds at an angle of 40° to the horizontal and a force of 20 pounds at an angle of 70° to the horizontal. Find the magnitude and the direction of the resultant. Is there more than one solution depending on the interpretation? Answer the question for all interpretations.

    (b) A boy pushes a lawn mower with a force of 50 pounds applied to the handle and along the handle which is 35° to the horizontal. What is the horizontal force moving the mower forward? What is the vertical force?
    Hint! Make a rectangle with the horizontal and vertical forces

15. Given $\vec{u} = (3,1), \vec{v} = (-2,2), \vec{w} = (1,3)$
    (a) find $(\vec{u} + \vec{v}) + \vec{w}$
    (b) find $\vec{u} + (\vec{v} + \vec{w})$
    (c) Do you think the associative law for addition of vectors holds?

16. Given $\vec{m} = (3,2), \vec{n} = (1,5)$:
    (a) find $2(\vec{m} + \vec{n})$;
    (b) find $2\vec{m} + 2\vec{n}$.
    (c) Do you think scalar multiplication is distributive over vector addition?
    (d) Represent parts (a) and (b) by diagrams.

17. The Cartesian representation of the velocity of an airplane is the vector (3,4). The wind velocity is represented by a vector (5,2). Find the vector which represents the final velocity of the plane. If each unit represents 100 miles per hour what is the final velocity of the plane. This is called the resultant velocity.

18. An airplane flies northwest at 300 miles per hour in still air. There is a wind blowing from the northwest at 50 miles per hour. What is the speed and direction of the plane in the wind? Use a diagram for a solution.

7·2 Using Vectors to Solve Problems in Statics

Vectors were introduced into mathematics through very practical applications in physics. In science the concept of a vector dealt with the magnitude and direction of forces, displacements, velocities, and accelerations. The algebraic development of vectors came later as an ordered n-tuple of real numbers.

One branch of science is known as *mechanics*. It is a combination of physics and mathematics. Mechanics can be divided into two areas:

(a) dynamics—the study of moving objects under forces.

(b) statics—the study of forces in equilibrium.

Hence, in dynamics the forces are unbalanced so as to produce motion whereas in statics the forces are balanced so that there is no motion.

Forces can be represented by vectors. If the forces are working in the same or opposite directions we can represent them by vectors in one dimension (or in 1-space); if the forces are working in two or more directions in a plane we can represent them by vectors in two dimensions (or in 2-space); if the forces are working in three dimensions we can represent them by vectors in three dimensions (or in 3-space).

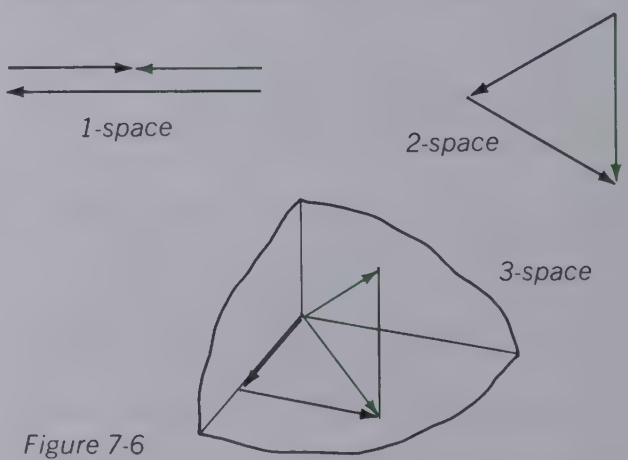


Figure 7-6

An excellent way to illustrate a simple statics problem is illustrated in Project 1.

Projects

Project 1

Purpose

To illustrate a vector solution to a problem in statics.

Method

1. Set up apparatus as shown in Figure 1.

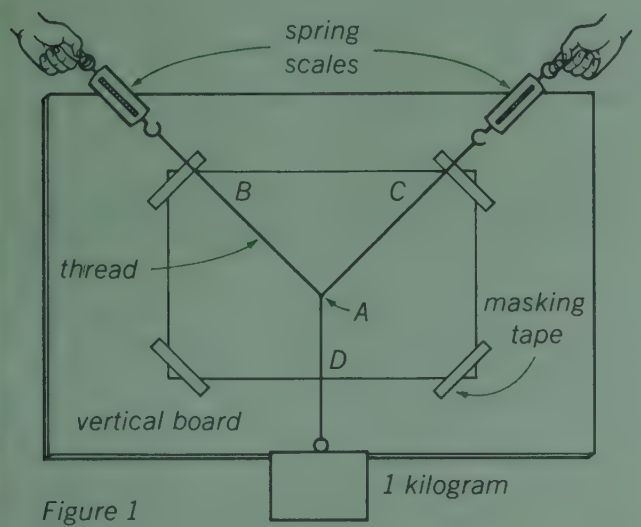


Figure 1

2. Hold the spring scales steady while your partner marks point A and the directions of $\vec{AB}$, $\vec{AC}$, $\vec{AD}$.
3. Record the readings on the scales.
4. Remove the paper and draw to scale vectors to represent the three forces.

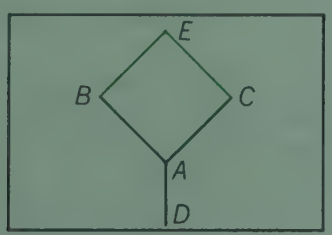


Figure 2

5. Complete the diagram to form a parallelogram $ABEC$ with two forces AB and AC.
6. Draw in the diagonal AE.
7. Compare the magnitude and direction of the diagonal AE and the third vector AD. What do you notice?
8. What *one* force could replace the force exerted by the two spring scales at B and C?
9. What can you say about the resultant of the two forces AB and AC? Discuss.
10. Compare the results of this project with the Law of Parallelogram of Vectors.

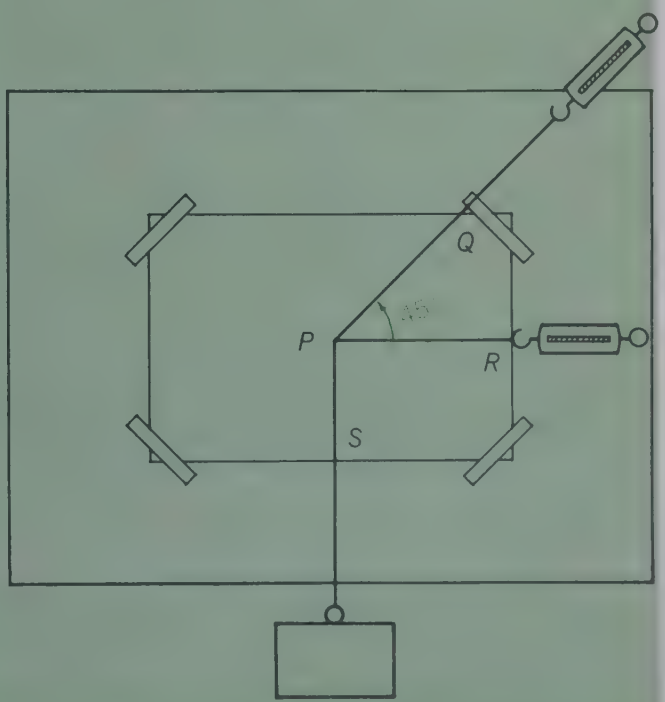
Project 2

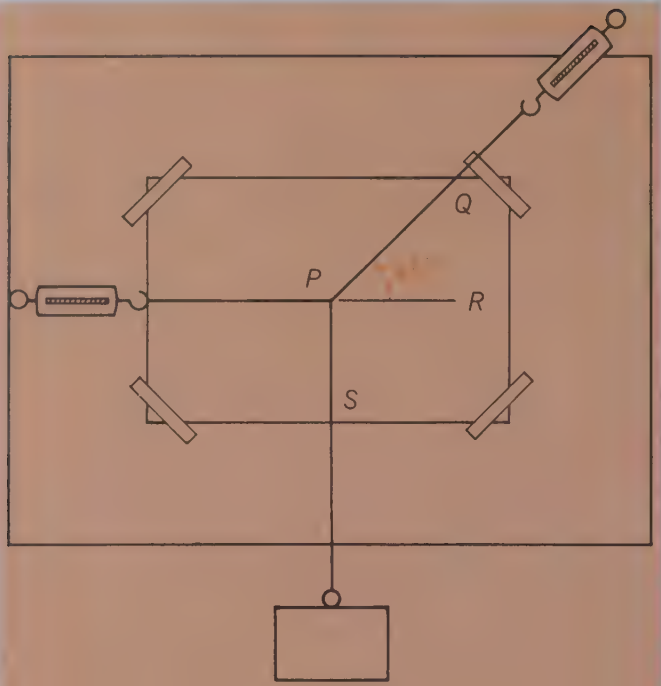
Purpose

To illustrate the Triangle Law of Vectors applied to forces in equilibrium.

Method

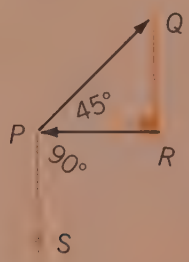
1. Use the vertical board of Project 1. Set up the apparatus as shown. Use a *pressure scale* (compression scale). If a pressure scale is not available use a second spring scale. (A push force can be replaced by a pull force with no change in the situation.)





1 kilogram

2. Hold steady while your partner marks point P and the directions for forces $\vec{PR}$, $\vec{PQ}$, $\vec{PS}$.
3. Record the readings on the scales.
4. Remove the paper and draw to scale a triangle of forces.

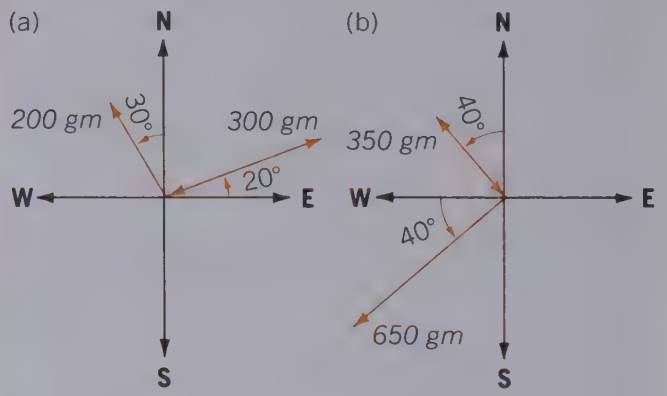


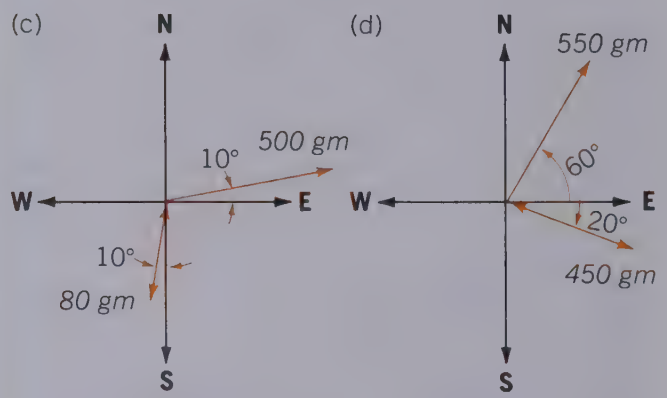
Notice that the initial end of $\vec{PQ}$ is the endpoint of $\vec{RP}$. Also $\vec{PS}$ can be represented by $\vec{QR}$. $\vec{QR}$ is equivalent to $\vec{PS}$.

5. What do you notice about the results? Is $\vec{PS} = \vec{QR}$? Is $\vec{PS}$ in the same direction as $\vec{QR}$?
6. Compare these results with the Triangle Law of Vectors.
   What similarities are there?
   Are there any differences?

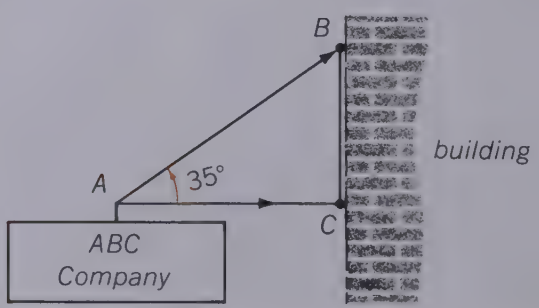
EXERCISE 7·2

1. What does *in equilibrium* mean?
2. Find geometrically the sum of the following pairs of forces.

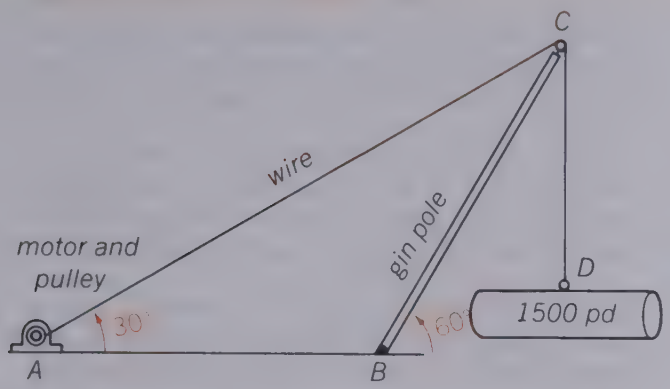




3. Find geometrically the tension (force) in the wire AB that holds the sign that weighs 150 pounds. Find the pressure on the rod AC.

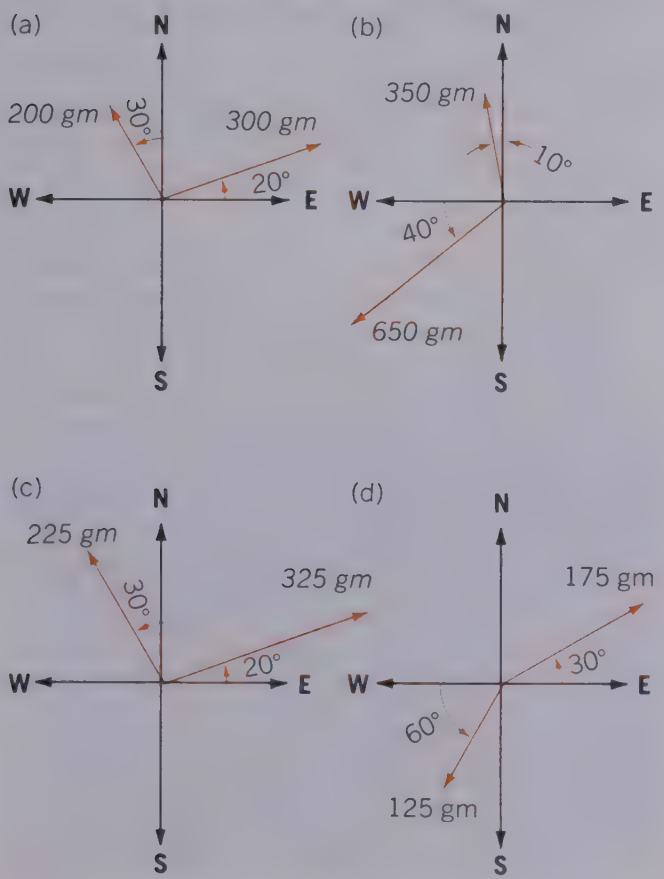


4. Find geometrically the tension or forces acting at *B*. *Hint!* Find the magnitude of the vectors *BA* and *CB*. Draw a vector triangle.



Loggers often use this arrangement to load logs onto trucks. This arrangement is called a GIN-POLE.

5. Find geometrically the one force that would balance the two forces in each case.

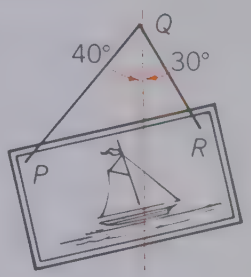


7·3 Resolution of Forces

We have seen how two or more vectors can be composed into a single resultant. Now let us look at how we can *resolve* a single vector force into a number of *components* (vectors) acting in specified directions. Notice the first example is in statics while the second is in dynamics.

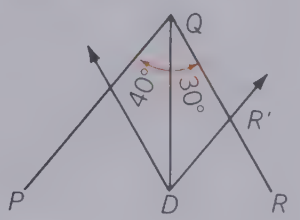
Example 1

A picture which weighs 18 pounds is suspended on a hook *Q* as shown. The two wires *QR* and *QP* form angles with the vertical of 30° and 40° respectively. Find the tension in each wire.



Solution

1. Draw a vertical line to represent the weight of the picture 18 units long. Label *Q*,*D*.



2. Draw $\overline{QR}$ such that $m\angle RQD = 30°$.
3. Draw $\overline{QP}$ such that $m\angle PQD = 40°$.
4. Through *D* draw a ray parallel to $\overline{QR}$ to cut $\overline{QP}$ at *P′*.
5. Through *D* draw a ray parallel to $\overline{QP}$ to cut $\overline{QR}$ at *R′*.
6. Measure $\overline{QP'}$ and $\overline{QR'}$. These measures represent the tensions in wires $\overline{QP}$ and $\overline{QR}$.
   Therefore, tension in $\overline{QP}$ is $9\frac{1}{2}$ pounds and the tension in $\overline{QR}$ is $12\frac{1}{2}$ pounds.

Note! We knew the single vector represented by $\vec{QD}$ to be the weight of the picture. We *resolved* this single vector into two other vectors represented by $\vec{QP'}$ and $\vec{QR'}$.

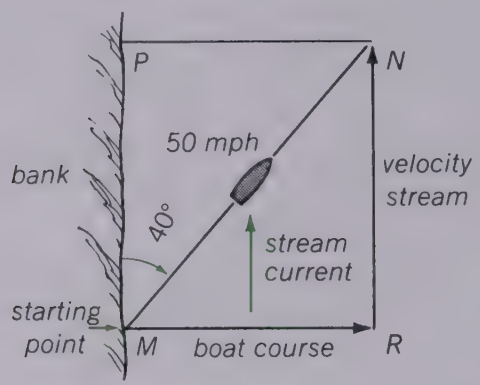
We should solve problems by trigonometry when we can. The following problem illustrates the resolution of forces by trigonometry.

Example 2

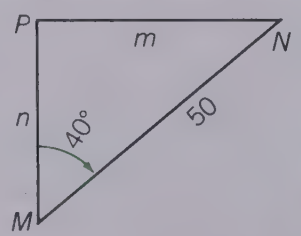
A motorboat sets course directly across a stream. The stream carries the boat downstream. The track is 40° with the bank and is 50 miles per hour. Find the course speed of the boat and the speed of the stream.

Solution

1. Draw a line to represent the bank and measure $\vec{MN}$ to represent 50 miles per hour at 40° to the bank as shown.
2. Through N draw a vector PN perpendicular to the bank.
3. Complete the parallelogram as shown.



4. Calculate the magnitude of $\vec{PN}$.
   This represents the velocity of the boat in still water (course speed).
   Solve by trigonometry:



$$\sin 40° = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$0.6428 = \frac{m}{50}$$

$$m = 32.14$$

   The course speed is 32.1 miles per hour (nearest tenth).

5. Calculate the magnitude of $\vec{RN}$. This represents the velocity of the stream.

$$\cos 40° = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$0.7660 = \frac{m}{50}$$

$$n = 38.3$$

   The speed of the stream is 38.3 miles per hour.

EXERCISE 7·3

1. A picture of 12 pounds hangs askew on the wall. The short wire forms an angle of 55° with the vertical while the long wire forms an angle of 30° with the vertical. Find the tension in each wire.

2. A picture weighing 15 pounds hangs askew on the wall. The short wire forms an angle of 45° with the vertical while the longer wire forms an angle of 35° with the vertical. Find the tension in each wire.

3. A motorboat sets course directly across a stream with course speed of 26 miles per hour. The stream carried the boat downstream at 16 miles per hour. Represent this by a vector diagram and find the resultant track speed and direction of the motorboat relative to the bank. Solve by trigonometry.

4. A child weighing 50 pounds in a swing is pushed by a horizontal force of 20 pounds. Find the tension in the swing rope by trigonometry.
   Hint! The child's weight acts straight down.

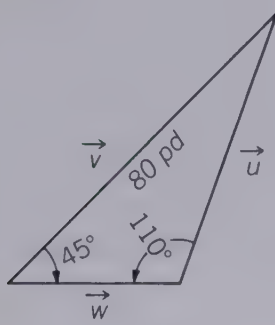
5. A boy on a swing is pushed by a horizontal force. The tension in the rope is 78 pounds when the rope forms an angle of 33° with the vertical. Resolve the 78 pound force into a horizontal and a vertical force.

6. A pilot wishes to fly his aircraft from Windsor to Toronto. Toronto is N65°E of Windsor. The air speed of the plane is 210 miles per hour. There is a wind blowing due north at 35 miles per hour. Find the course the pilot must set in order to arrive in Toronto. Find the ground speed of the aircraft.

7·4 The Sine Law

We can use trigonometry in many ways to solve problems in statics and dynamics. Study the two examples that follow.

Example 1



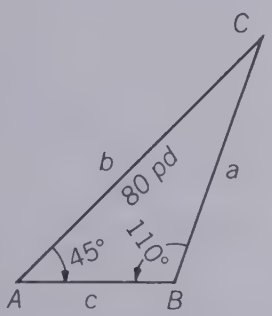
Three forces represented by $\vec{u}, \vec{v}$ and $\vec{w}$ are in equilibrium as shown. The force represented by $\vec{v}$ is 80 pounds. Find the forces represented by $\vec{u}$ and $\vec{w}$.

Solution

Recall the Sine Law:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Label the diagram using a, b, c, A, B, C.



Now find $m\angle C$.

$m\angle C + 45° + 110° = 180°$

$m\angle C = 25°$

Now choose any two ratios such that b ($\vec{v}$ has a force of 80 pounds) is involved.

(1) $\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$

(2) Make replacements: $\dfrac{\sin 110°}{80} = \dfrac{\sin 45°}{a}$

Remember: $\sin(180° - \alpha°) = \sin \alpha°$; hence

$\sin(180° - 70°) = \sin 70°$

(3) Therefore: $\dfrac{\sin 70°}{80} = \dfrac{\sin 45°}{a}$

(4) Using tables: $\dfrac{.9397}{80} = \dfrac{.7071}{a}$

(5) $a = \dfrac{.7071 \times 80}{.9397}$

$a = 60$ *slide rule accuracy*

Can you use your slide rule to solve this directly using equation (3)?

(6) Solving for c.

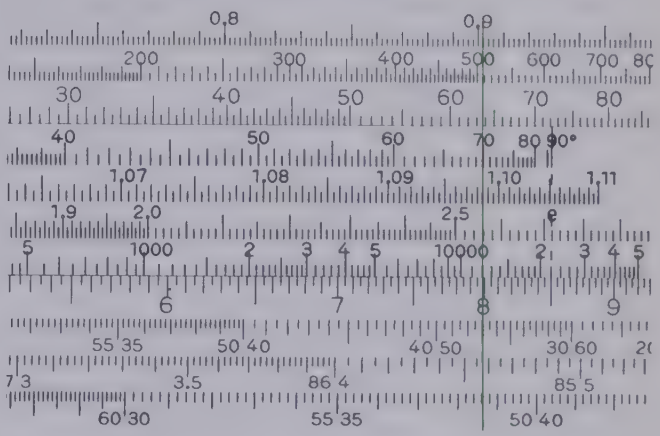
$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

(7) $$\frac{\sin 110°}{80} = \frac{\sin 25°}{c}$$

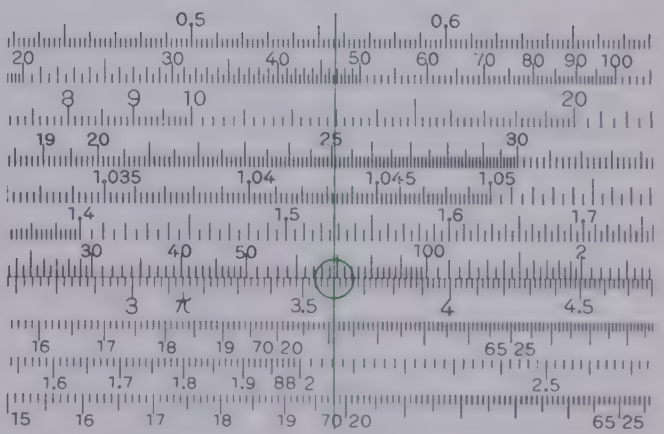
$c = 36$ *slide rule*

The two forces are 60 pounds and 36 pounds. A slide rule solution using equation (7) is shown here.

(a) Place hairline on 8 on D scale. Move slide so that 70 on S scale is on hairline.



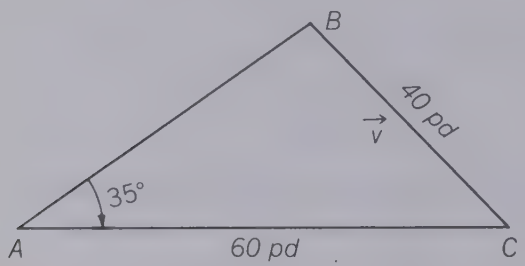
(b) Move hairline so that it is over the 25 on S scale. Read answer on D scale.



In a similar way we can find the direction of the forces in certain problems – using the Sine Law.

Example 2

Find the angles between the forces given the information in the triangle of forces below. The forces are in equilibrium.



Solution

We will find the measure of $\angle B$ then calculate the measure of $\angle C$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 35}{40} = \frac{\sin B}{60}$$

Using the slide rule or tables we get

$$m\angle B = 59°$$

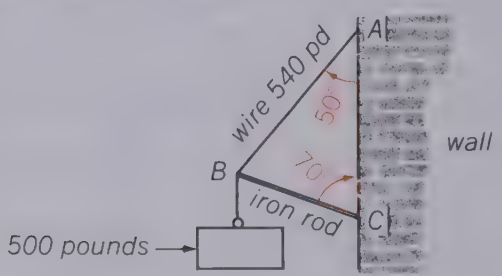
To find $m\angle C$ we use

$$m\angle C + 35° + 59° = 180°.$$

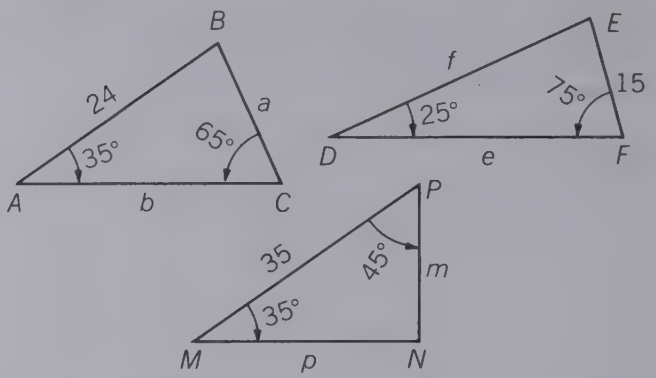
Calculate $m\angle C$.

EXERCISE 7·4

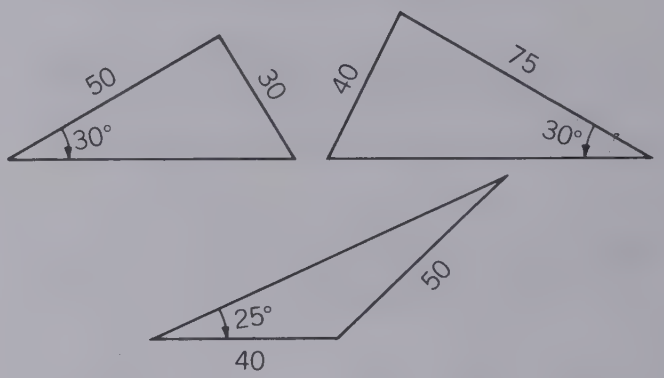
1. Calculate the measure of $\angle B$ and $\angle A$. Then calculate the force exerted by the iron rod.



2. Use a slide rule (or trigonometric tables if you do not have a slide rule) to find the length of the unknown sides.



3. Find the angle between the two given sides.

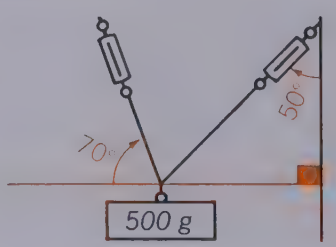


4. Calculate the third side of each triangle in question 3.

5. Solve, using trigonometry, question 4 Exercise 7·2.

6. Solve, using trigonometry, question 3 Exercise 7·2. Did one setting of your slide rule give you both answers?

7. Use the apparatus shown here to check your algebra in question 1.



7·5 The Cosine Law

The Cosine Law will help us solve algebraically problems in statics and dynamics. Do you remember the three versions of the Cosine Law?

$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

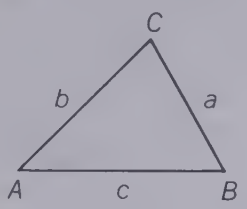
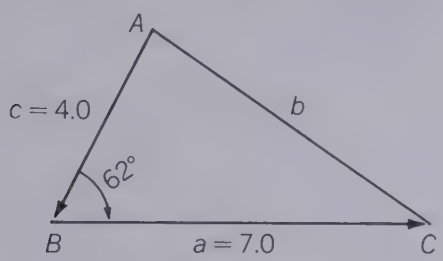


Figure 7-7

Discuss interesting ways of remembering the Cosine Law.

Example 1

Two forces, 4.0 pounds and 7.0 pounds, act in directions shown in the diagram. Calculate the third force (b) which will act in equilibrium with the other forces.



Solution

We will use the version of the Cosine Law that involves angle B since it is the only angle we know; also in this case b will be given.

$b^2 = a^2 + c^2 - 2ac \cos B$
$b^2 = 7^2 + 4^2 - 2 \cdot 7 \cdot 4 \cos 62°$ *replacements*
$b^2 = 49 + 16 - 56 \times .4695$ *tables*
$b^2 = 65 - 26.3$
$b^2 = 38.7$
$b = 6.2$ *2 significant figures*

The third force is 6.2 pounds.

We can use the Sine Law to find the other angles if it is necessary. This will be left to the student. The measure of $\angle C$ is 39° and of $\angle A$ is 79°. *Hint!* Find the smaller angle first

When working with the Cosine Law you must remember how to express the cosine of an obtuse angle in terms of an acute angle. Do you remember how to do this?

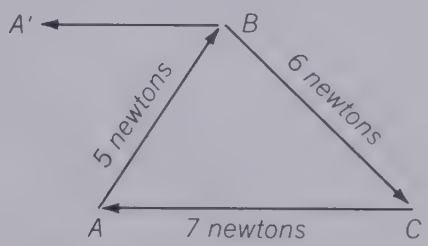
$$\cos 120° = \cos (180° - 60°)$$
$$= - \cos 60°$$

Now try these two: cos 150° = ?
cos 123° = ?

Did you get – cos 30° and – cos 57°?

Example 2

Three magnetic forces act in Point B. The forces are represented by $\vec{BA'}$, $\vec{BC}$, $\vec{AB}$. $\vec{BA'}$ is equivalent to $\vec{CA}$. Calculate the angles between the three forces acting in equilibrium.



Solution

Use the Cosine Law and solve for the *smallest* angle first (it is opposite the smallest side).

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$25 = 36 + 49 - 84 \cos C$$
$$-60 = -84 \cos C$$
$$\cos C = \tfrac{60}{84} = .714$$
$$C = 44°$$

We now use the Sine Law to find the next larger angle:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

Do you get $m\angle A = 57°$?

EXERCISE 7·5

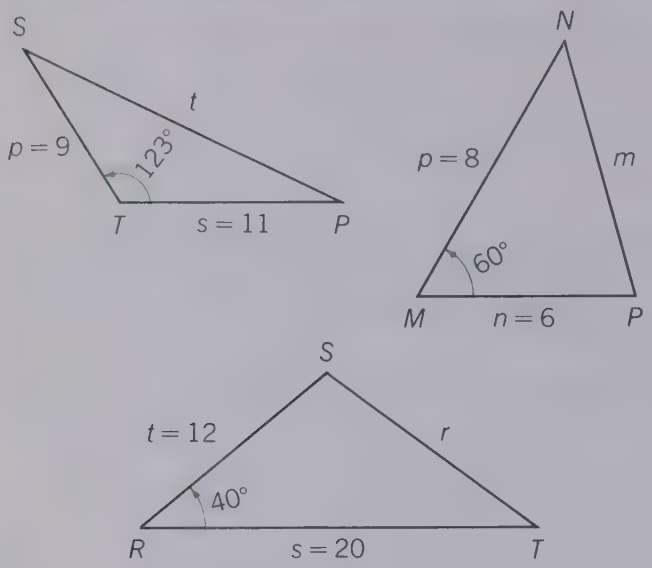
1. Express each as the cosine of an acute angle.

   (a) cos 100° (b) cos 147° (c) cos 109°

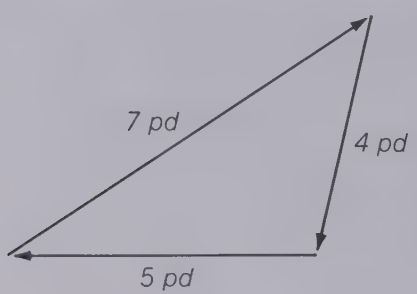
2. Find the value of each:

   (a) cos 112° (b) cos 137° (c) cos 162°
   (d) sin 100° (e) sin 147° (f) sin 143°

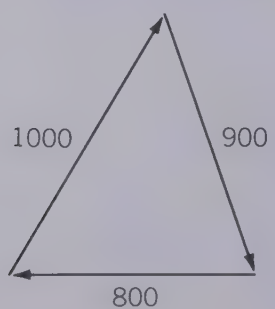
3. Solve the following triangles by calculating the side opposite the given angle first.



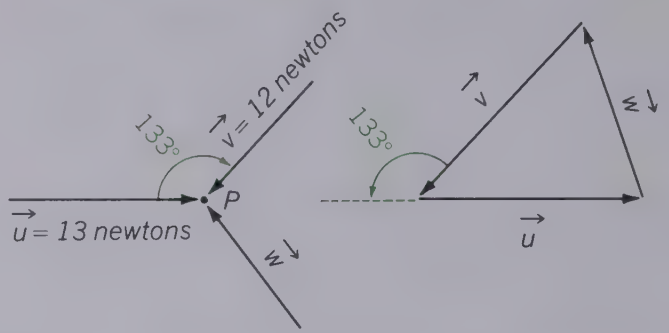
4. Find the angles between the forces in equilibrium.



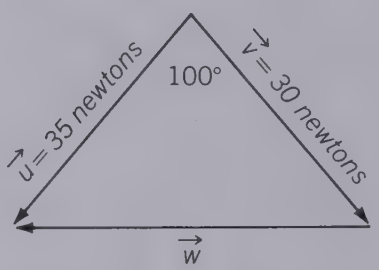
5. Calculate the angles between the forces in equilibrium.



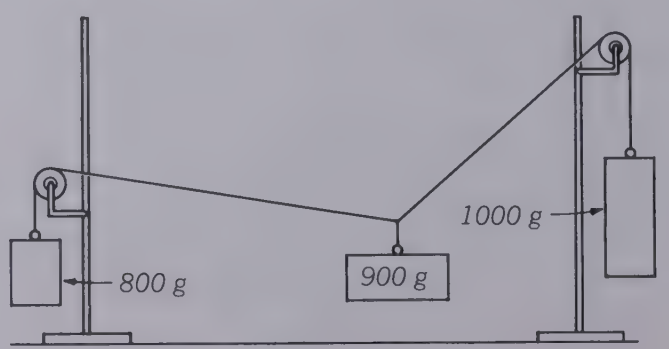
6. Three magnetic forces, one of 12 newtons, one of 13 newtons and one unknown, are acting in equilibrium on a particle *P*. These forces are represented by $\vec{v}$, $\vec{u}$ and $\vec{w}$ respectively. The directions are shown in the diagram. Find the magnitude of $\vec{w}$. Notice how the vectors have been drawn in the second diagram so that they fit the model for the Triangle Law of Vectors.



7. Calculate the third force in newtons and its direction with respect to the other forces given the information in the diagram and that the three forces are in equilibrium.



8. Verify question 5 experimentally by using the apparatus shown here or some other suitable apparatus.



7·6 Moments of Force

Certain problems in statics are handled by the Principle of Moments. We need to know the meaning of *moments of force.*

> The moment of force about a fixed point is the product of the force itself and the perpendicular distance from the line of action to the fixed point of rotation.

Consider the following:

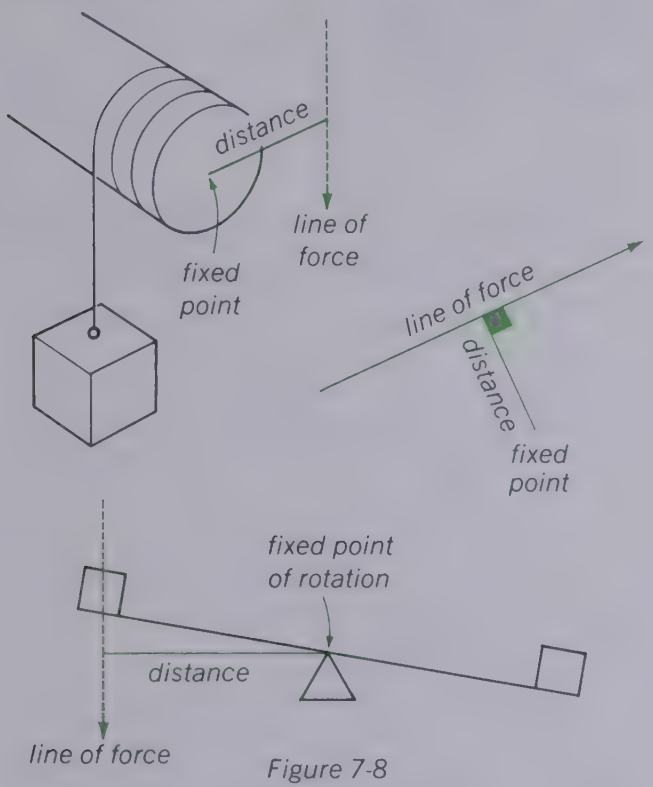


Figure 7-8

The moments of force is a measure of the *turning power* of the forces about the point. One unit of measure of moment of force is obtained by looking at the product.

Moment of force = force × distance
= force in pounds × distance in feet
= _____foot-pounds.

Can you name other units for moments of force?

Principle of Moments

> When an object, free to move about a fixed point, is in equilibrium, the sum of all the clockwise moments is equal to the sum of all the counterclockwise moments.

Consider a lever with the forces acting as shown. We will consider the lever to be weightless at this time.

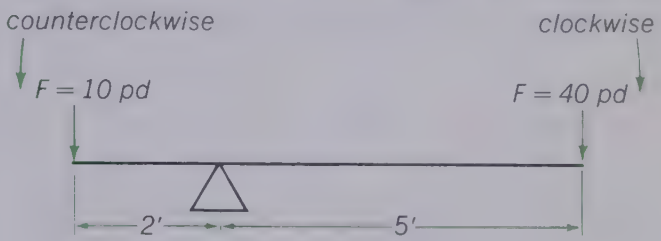


Figure 7-9

Consider:

The Clockwise moments

$$F \times D = 4 \text{ pd} \times 5 \text{ ft}$$
$$= 20 \text{ ft-pd.}$$

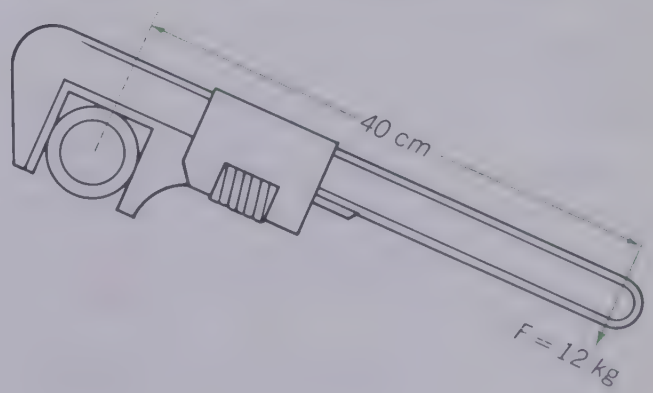
Counterclockwise moments

$$F \times D = 10 \text{ pd} \times 2 \text{ ft}$$
$$= 20 \text{ ft-pd.}$$

Since the clockwise moments are equal to the counterclockwise moments the system is in equilibrium.

Example 1

Calculate the moment of force resisting the force of the pipe wrench. The system is in equilibrium.



Solution

Since the system is in equilibrium we know

clockwise moments = counterclockwise moments

$F \times D = M$ *this represents the moment of force*

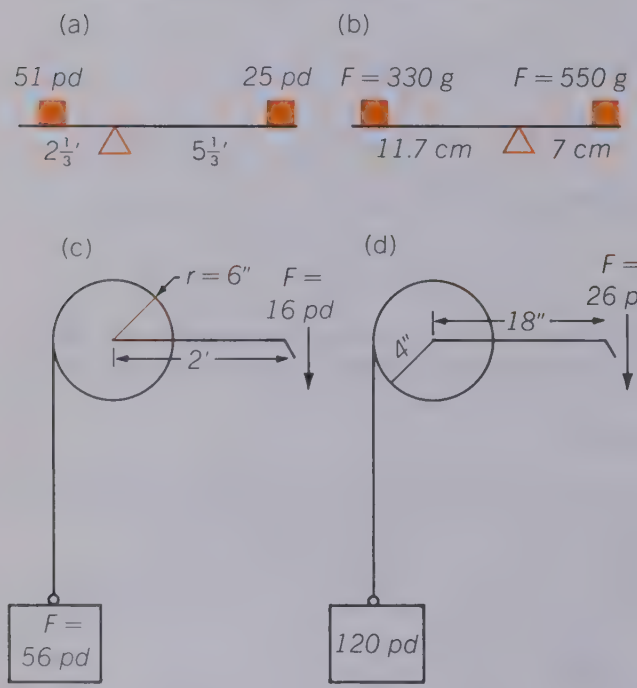
$12.000 \text{ g} \times 40 \text{ cm} = M$

$480.000 \text{ g–cm} = M$

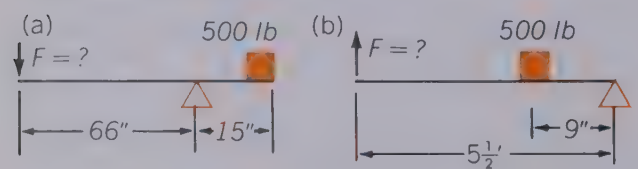
The counterclockwise moment resisting the pipe wrench is 480,000 g–cm.

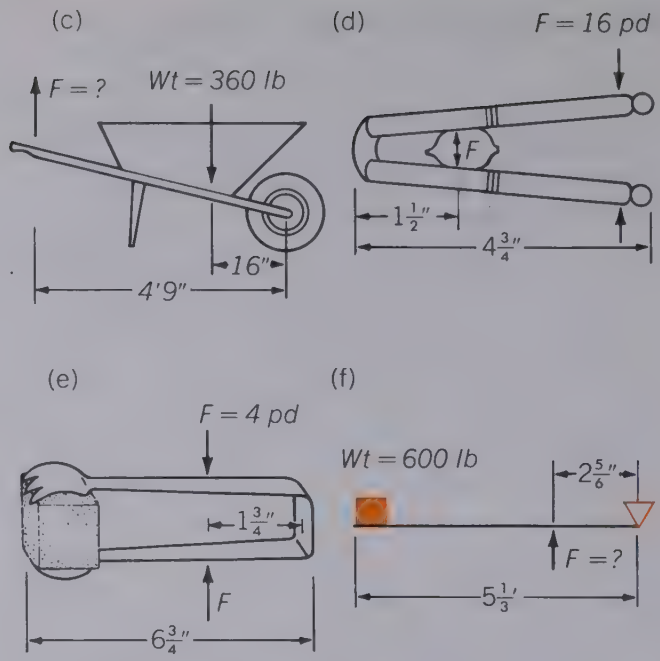
EXERCISE 7·6

1. Determine whether or not each system is in equilibrium. What additional forces must be applied to obtain equilibrium for those not in equilibrium?

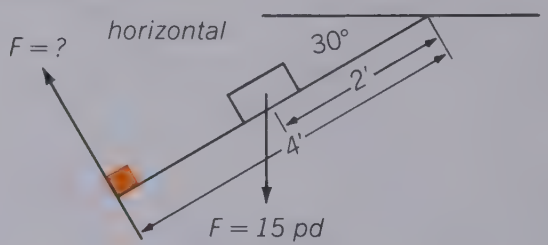


2. Calculate the force necessary to create a state of equilibrium in each case.

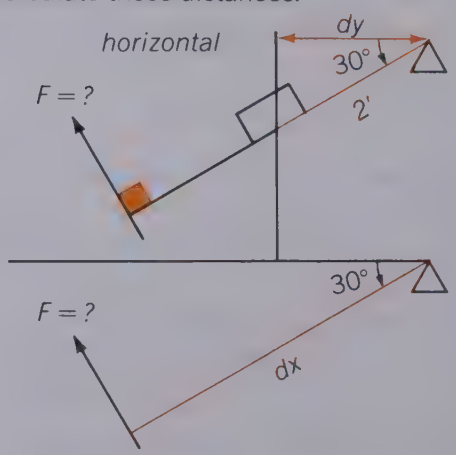




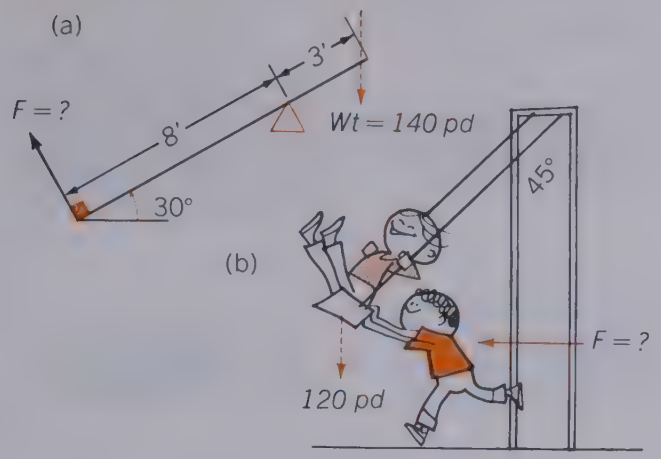
3. Calculate the indicated force, F, needed to locate a state of equilibrium. Some hints are given.



Hint! Draw in the perpendicular distance from the lines of force to the point of rotation – the fulcrum. Then calculate these distances.



4. Calculate the force, F, required to establish a state of equilibrium.



7·7 Mechanical Advantage and Heavy Levers

While there are a number of different formulas and definitions for mechanical advantage, we shall make this statement.

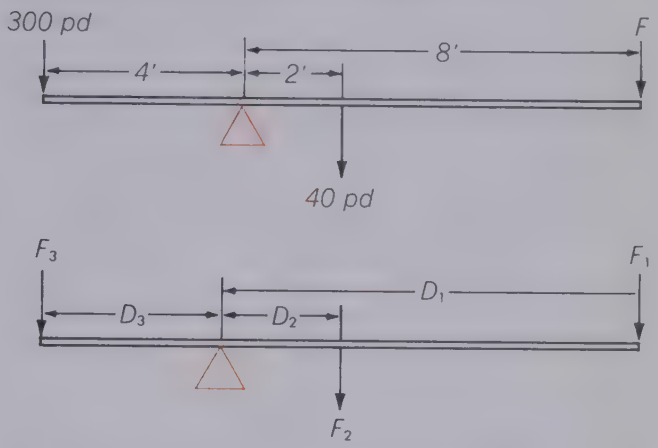
> Mechanical advantage of a machine is expressed by the ratio
>
> $$\frac{\text{resistance overcome } (R)}{\text{effort applied } (E)}.$$
>
> $$\text{M.A.} = \frac{R}{E}$$

In the previous section we considered our levers to be weightless. In this section we shall consider our levers to have weight and hence they will affect our calculations.

The *centre of gravity* of an object is defined as the point through which we consider all the weight of the object to act. It is considered as though the total weight of the object were located at that point – the centre of gravity. In a regular shaped object the centre of gravity is located at the centre.

Example 1

A heavy lever of uniform shape 12 feet long and weighing 40 pounds is used to lift an object weighing 300 pounds. The distance from the weight to the fulcrum is 4 feet. What force applied at the end of the lever is necessary to lift the weight?



Consider the force necessary to lift the object is the force necessary to establish an equilibrium.

Calculate the Mechanical Advantage of the system.

Solution

1. The weight of the heavy lever is located at the centre of the lever. Hence the 40 pounds acts as a point 6 feet from the end. Hence in this case the weight of the lever helps lift the weight.

2. Calculate the force needed.

   clockwise moments = counterclockwise moments

   $$(F_1 \times D_1) + (F_2 \times D_2) = F_3 \times D_3$$
   $$(F_1 \text{ pd} \times 8 \text{ ft}) + (40 \text{ pd} \times 2 \text{ ft}) = 300 \text{ pd} \times 4 \text{ ft}$$
   $$8F_1 \text{ ft-pd} = 1120 \text{ ft-pd}$$
   $$F_1 \text{ pd} = \frac{1120 \text{ ft-pd}}{8 \text{ ft}}$$
   $$F_1 \text{ pd} = 140 \text{ pd}$$

   The force necessary is 140 pounds

3. Calculate the *M.A.*

   $$M.A. = \frac{\text{resistance overcome } (R)}{\text{effort applied } (E)}$$
   $$= \frac{300}{140}$$
   $$= 2.14$$

   The mechanical advantage is 2.14.

Another term sometimes used for *M.A.* (when the lever is weightless or the machine is considered perfect) is velocity ratio. Velocity ratio is also used in calculating efficiency of a machine.

$$\text{Velocity ratio} = \frac{\text{distance effort moves } (De)}{\text{distance resistance moves } (Dr)}$$

$$V.R. = \frac{De}{Dr}$$

Efficiency of a machine is defined by

$$\text{Efficiency} = \frac{M.A.}{V.R.} = \frac{\frac{R}{E}}{\frac{De}{Dr}} = \frac{R}{E} \times \frac{Dr}{Re} = \frac{R \times Dr}{E \times Re}$$

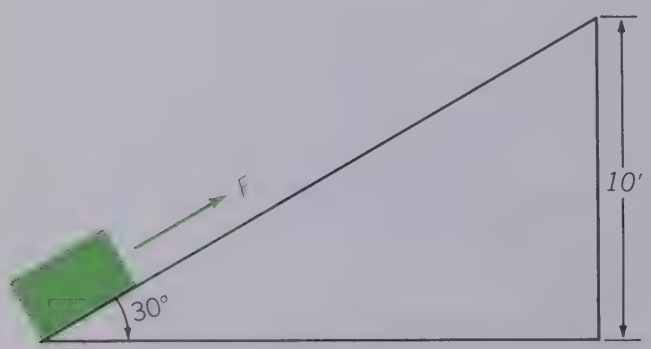
Since efficiency is usually expressed in a percent we write

$$\text{Efficiency} = \frac{R \times Dr}{E \times Re} \times 100\%.$$

Sometimes $R \times Dr$ is called "work got out of machine" and $E \times De$ is called "work put into machine".

Example 2

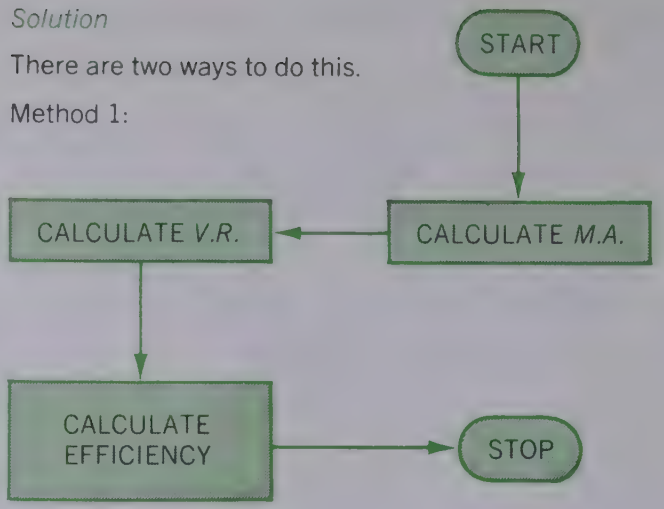
An inclined plane is used to raise a 400 pound weight 10 feet (vertical height). A force of 300 pounds is needed. Calculate the efficiency of the incline plane.



Solution

There are two ways to do this.

Method 1:



(a) Calculate the *M.A.*

$$M.A. = \frac{\text{Resistance } R}{\text{Effort } E} = \frac{400}{300} = 1.33$$

(b) Calculate the *V.R.*

$$V.R. = \frac{De}{Dr}$$

The distance *Dr* is the height of the vertical incline plane.
The distance *De* is the slant height of the incline plane.

$$\sin 30° = \frac{10}{De}$$

$$De = \frac{10}{\sin 30°}$$

$$= 10 \times \frac{1}{\sin 30°} \left(\frac{1}{\sin 30°} = \csc 30°\right)$$

$$= 10 \times \csc 30°$$

$$= 10 \times 2.00$$

$$De = 20$$

$$V.R. = \frac{20}{10}$$

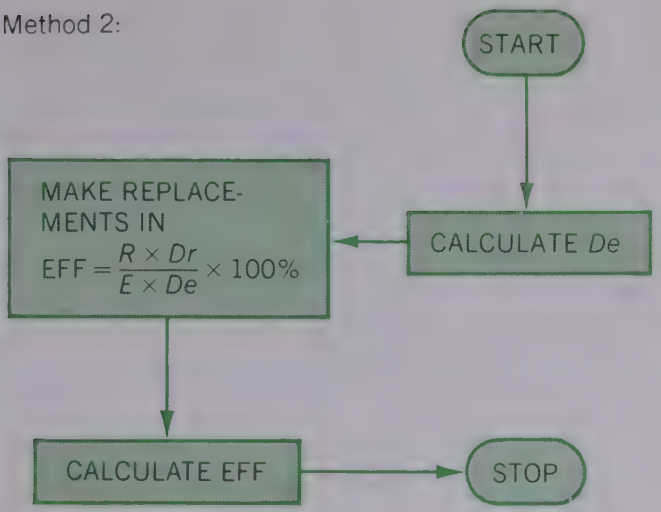
$$= 2$$

(c) Use $\text{Efficiency} = \frac{M.A.}{V.R.} \times 100\%$

$$= \frac{1.33}{2} \times 100\%$$

$$= 66.7\%$$

Method 2:



(a) We will use the formula

$$\text{Efficiency} = \frac{R \times Dr}{E \times De} \times 100\%$$

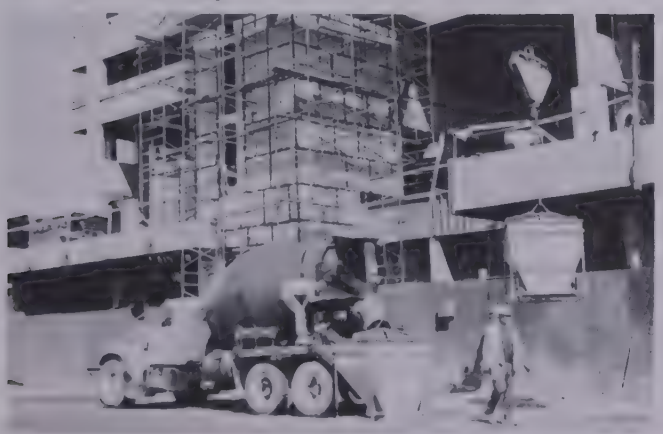
We know $R = 400$ pd
$Dr = 10$ feet
$E = 300$ pd
$De =$ unknown

(b) We must find De as we did in step (b) of Method 1.

$De = 20$ feet

(c) $$\text{Efficiency} = \frac{R \times Dr}{E \times De} \times 100\%$$
$$= \frac{400 \times 10}{300 \times 20} \times 100\%$$
$$= 66.7\%$$

(d) The efficiency is 66.7%.



Use of the principles of mathematics can greatly reduce the difficulty of lifting very heavy objects.

EXERCISE 7·7

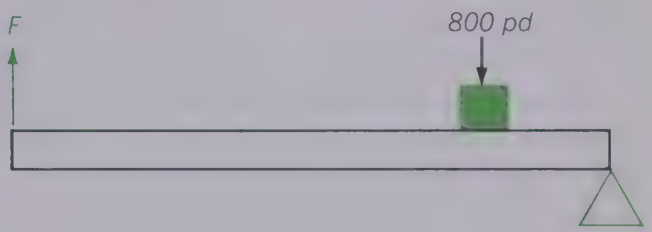
1. Calculate the force needed to lift the object in Example 1 if the lever had been weightless. Calculate the mechanical advantage.

2. Why do you think $R \times Dr$ is sometimes called "*the work got out of the machine*"?

3. Why do you think the product $E \times De$ is sometimes called "*the work put into the machine*"?

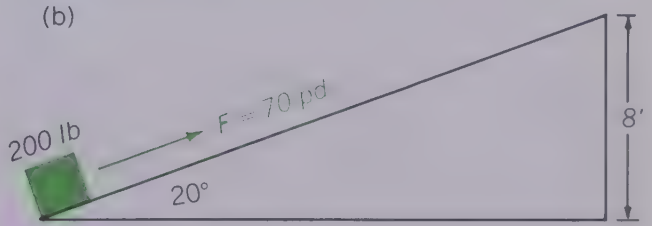
4. What factors do you think caused the incline plane (Example 2) to have an efficiency of less than 100%.

5. Calculate the M.A. of each system.
(a)



A heavy lever weighing 10 pounds is used to lift 800 pounds which is placed 2 feet from the fulcrum. The force is applied 8 feet from the fulcrum.

(b)

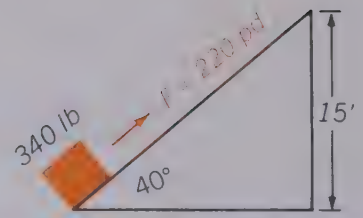


A 200 pound weight is pulled up an inclined plane as illustrated. A force of 70 pounds is required.

6. In question 5 the weight in (a) is lifted 1 foot. How far must the force move? Calculate the efficiency of the lever.

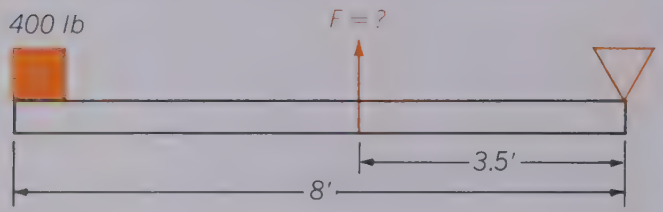
7. In question 5, part (b), calculate the efficiency of the incline plane.

8.



A 340 pound weight is raised a vertical distance of 15 feet on a 40° inclined plane by a force of 220 pounds. Calculate the efficiency.

9.



A heavy lever weighing 20 pounds is used to lift 400 pounds as illustrated. Calculate:

(a) the force required to lift the weight

(b) the distance the force moves if the weight moves 1 foot

(c) the M.A.

(d) the efficiency.

10. Can you design a system using a heavy lever 10 feet long and weighing 20 pounds that will lift a weight of 60 pounds without any additional force being needed? Draw a diagram to show your system.

Projects

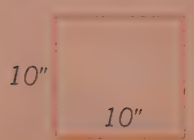
Project 1

Purpose

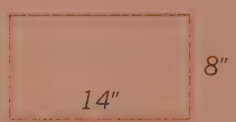
To find the centre of gravity of several plane shapes.

Method

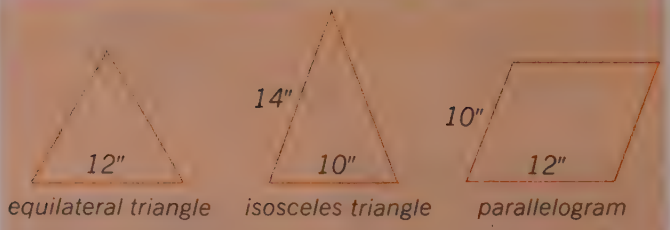
1. Obtain the following plane shapes cut from wood or heavy corrugated cardboard. Sizes are suggested.



square



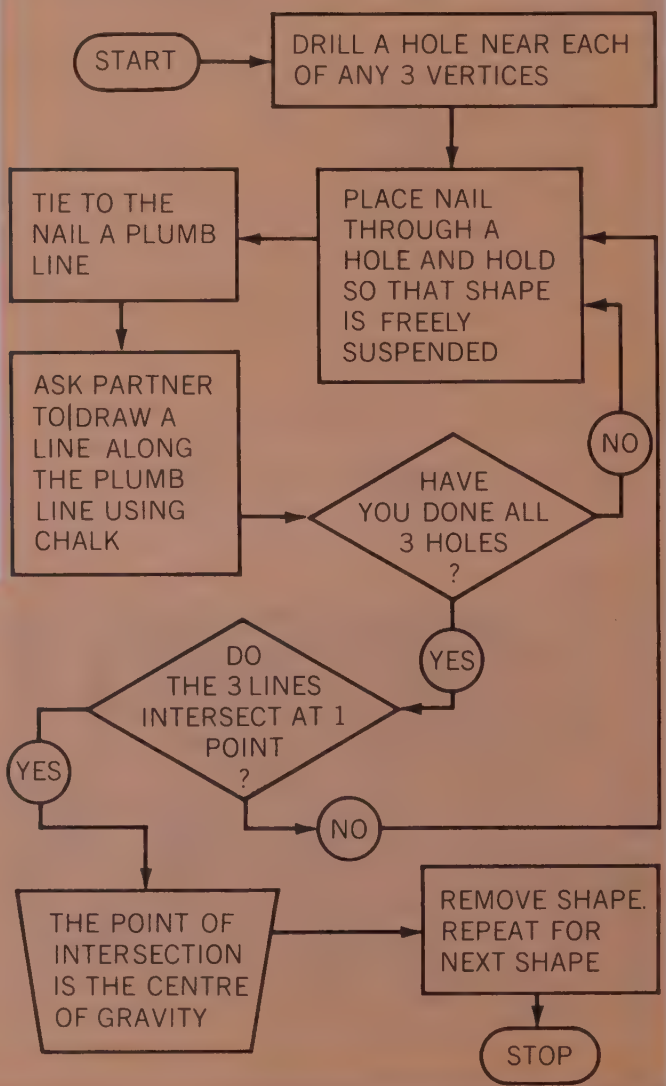
rectangle

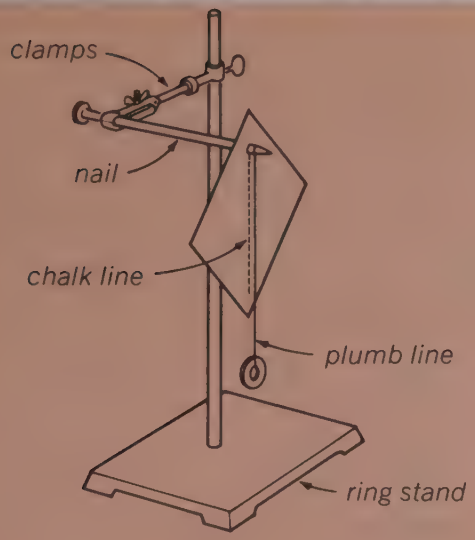


equilateral triangle *isosceles triangle* *parallelogram*

two irregular shapes *scalene triangle*

2. For each shape follow these steps:


START
DRILL A HOLE NEAR EACH OF ANY 3 VERTICES
PLACE NAIL THROUGH A HOLE AND HOLD SO THAT SHAPE IS FREELY SUSPENDED
TIE TO THE NAIL A PLUMB LINE
ASK PARTNER TO DRAW A LINE ALONG THE PLUMB LINE USING CHALK
HAVE YOU DONE ALL 3 HOLES ?
NO
YES
DO THE 3 LINES INTERSECT AT 1 POINT ?
YES
NO
THE POINT OF INTERSECTION IS THE CENTRE OF GRAVITY
REMOVE SHAPE. REPEAT FOR NEXT SHAPE
STOP



Questions

1. Can you balance each shape on a pencil tip by placing the point of intersection of the 3 lines on the tip?
2. Draw the diagonals of the square, rectangle and parallelogram. What do you notice about the centre of gravity and the intersection of the diagonals?
3. Draw the medians of the triangles. What do you notice about the intersection of the three medians and the centre of gravity?

Project 2

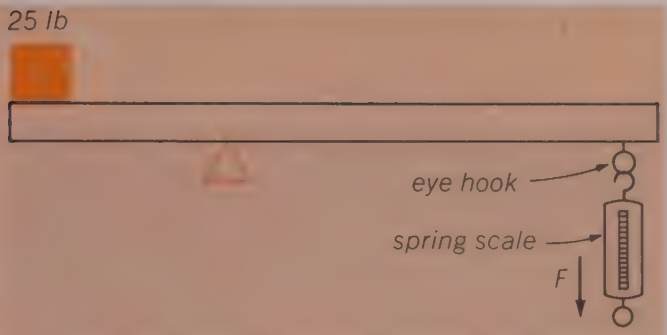
Purpose

To experiment using *Heavy Levers*.

Method

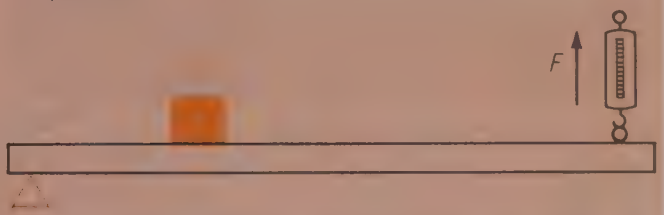
1. Obtain a length of 2 × 4 about 8′ or 10′ long for a lever and a short piece of 2 × 4 about 4″ long and shape it like this for a fulcrum. (A 2 × 4 is 2″ × 4″.)

2. Set up a lever system by which a weight of 25 pounds (or thereabouts) can be lifted by just the weight of the lever. Record *Dr*. Draw a diagram and show mathematically why it works.
3. Set up this system. The weight should be about 25 pounds.



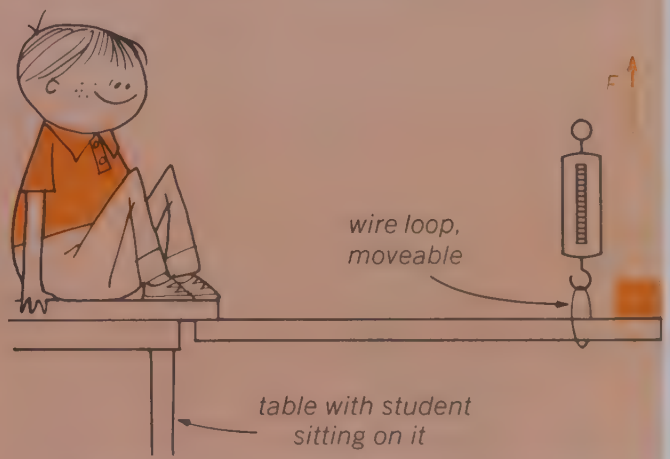
Make a number of recordings with the fulcrum in different positions; some with the fulcrum near the resistance, some with the fulcrum near the force. Calculate the *M.A.* for each.

4. Set up this system. Weight should be 10 to 25 pounds.



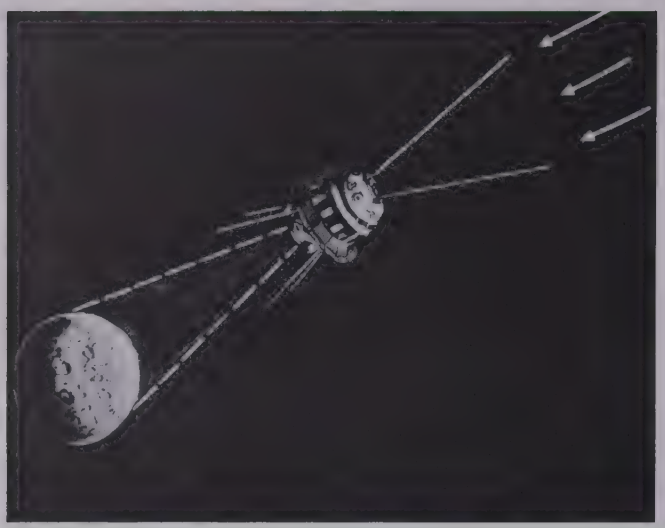
   (a) Make a number of recordings as you move the weight near the fulcrum and nearer the force.
   (b) Calculate the *M.A.* for each recording.
   (c) What do you notice?

5. Set up this system. Use a weight of about 5 pounds.



   (a) Make several recordings as you move the scale from near the table (fulcrum) to near the weight.
   (b) Calculate the *M.A.* for each.
   (c) What do you notice?

7·8 Vectors in 3-Dimensional Space



Project

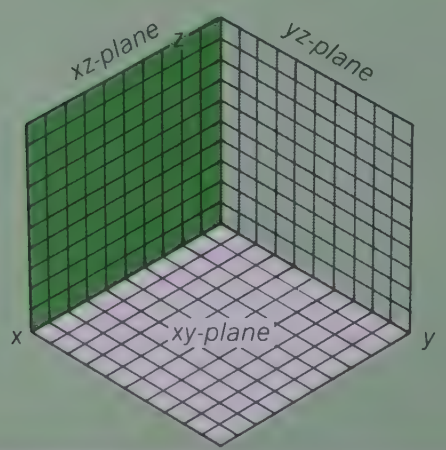
Project 1

Purpose

To illustrate vectors in 3-D space.

Method

1. Obtain a small cardboard box at least 13" × 13" × 13". (Corrugated cardboard box is the best.) Cut away 2 adjacent sides and the top. On each of the three remaining surfaces (1 bottom and 2 adjacent sides) draw (or glue) 1" grids. Label as shown.



2. In the *xy*-plane put a hole at (3,4). Through this hole place a plastic drinking straw perpendicular to the plane. Use Plasticine or clay to make it stand up.
3. In the *xz*-plane place a hole at (3,5) and repeat step 2.
4. In the *yz*-plane place a hole at (4,5) and repeat step 2.
5. All three straws should meet. Use a pipe cleaner, string or glue to tie all 3 straws together. Clip the loose ends from all straws.
6. Place a 4th straw (colored red) from the origin to the intersection of the 3 straws. Glue in place. This red straw represents the vector (3,4,5).
7. Repeat steps 2 to 6 for points (10,9) in the *xy*-plane, (10,13) in the *xz*-plane, and (9,13) in the *yz*-plane. Use a blue straw to represent the vector (10,9,13).
8. Place and glue a yellow (or third color) straw on the ends of the red and blue straws.

Questions

1. Write each of the 3-D vectors in this form:
   initial point endpoint
   (__,__,__), (__,__,__). There are three vectors represented by the red, blue and yellow straws. Remember to name the coordinates in this order, (*x*,*y*,*z*).
2. Can you write a vector equation using the 3 vectors?
3. Write the symbol for each of the vectors in standard position. For example, if we were given the vector [(2,3,4), (7,9,11)] we would write (5,6,7) since this vector indicates a change of 5 in the *x*-direction, 6 in the *y*-direction, and 7 in the *z*-direction.
4. Write a vector equation using the 3 vectors written in standard position.

Earlier we saw that a vector on a line could be represented by a single number (*x*). Later we saw that a vector in two-dimensional space could be represented by an ordered pair (*x*,*y*). In three dimensions how can we represent a vector? Look at Figure 7-10 which is diagrammatic of the model. Note the manner in which the axes are labelled *x*,*y* and *z*.

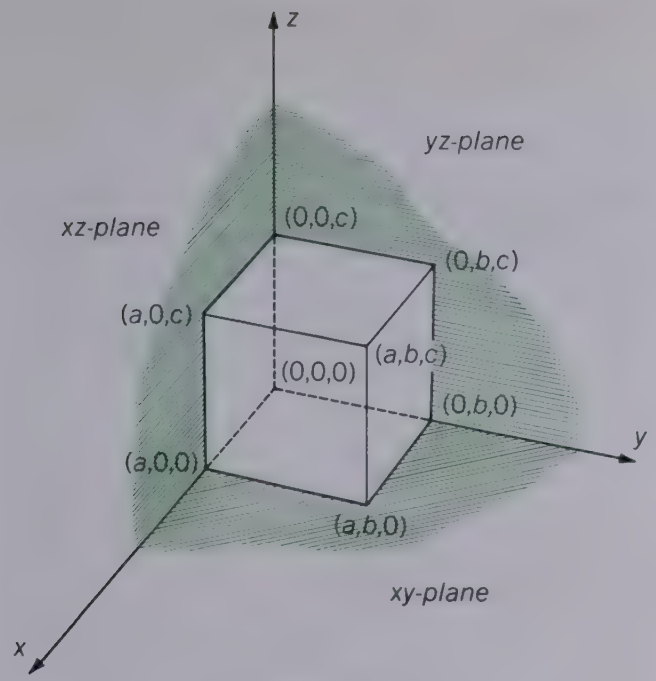


Figure 7-10

Since a point in space must be located in reference to each axis then there must be three components in the description: the *x*-component, the *y*-component and the *z*-component. If we refer to a rectangular box as in your model we need to identify a point by

(a) the distance it is from a wall

(b) the distance it is from an adjacent wall

(c) the distance it is from the bottom.

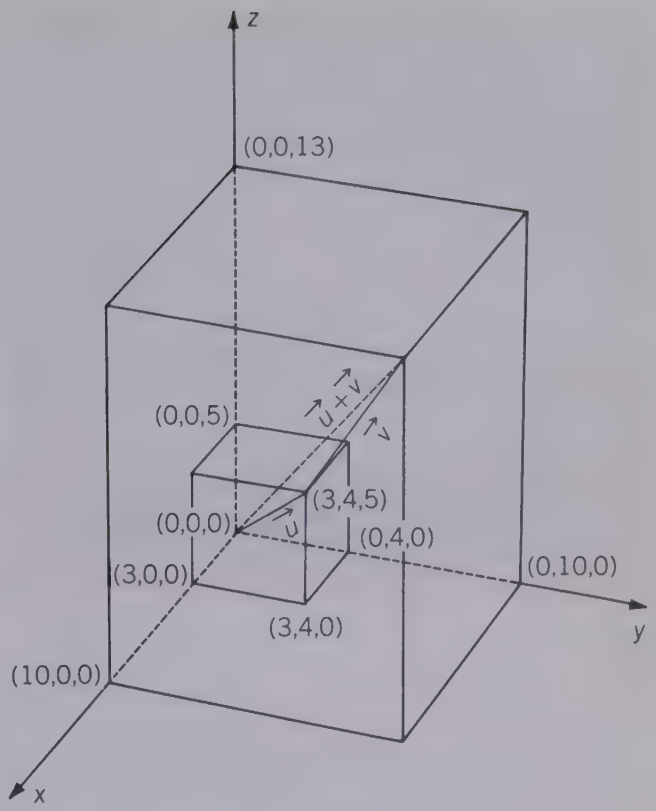
A vector in *standard position* (what does this mean?) could be described by an ordered triplet—(*a*,*b*,*c*) where *a* is the *x*-component, *b* is the *y*-component and *c* is the *z*-component.

Example 1

Find the sum of the two vectors $\vec{u} = (3,4,5)$ and $\vec{v} = (7,6,8)$.

Solution

In the diagram $\vec{v} = (7,6,8)$ was translated to the end of $\vec{u}$ so that [(0,0,0), (7,6,8)] became [(3,4,5), (10,10,13)]. Therefore the sum vector starts at (0,0,0) and ends at (10,10,13). The standard description then is $\vec{u} + \vec{v} = (10,10,13)$.



Since 3-dimensional diagrams are difficult to interpret let us try to find the sum by another means. Let us review the method of adding two vectors on a line then review adding two vectors in a plane. Then let's see if we can predict the manner in which to add two vectors in space.

On a line:

$(3) + (4) = 7 \quad (a) + (b) = (a+b)$

On a plane:

$(2,3) + (4,5) = (6,8) \quad (a,b) + (c,d) = (a+c, b+d)$

In space:

$(1,2,4) + (5,8,2) =$ **?**

Now don't read on until you have tried this:

$(a, b, c) + (d, e, f) =$ **?**

Did you have $(1,2,4) + (5,8,2) = (6,10,6)$?

Did you have $(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$?

We will use this generalization then to do the above problem.

$(3,4,5) + (7,6,8) = (10,10,13)$

This agrees with that solution we had earlier.

The properties we developed for vectors on a plane can be extended to vectors as a set.

(a) Multiplication by a scalar
$k(a,b,c) = (ka,kb,kc)$

(b) Subtraction

(c) The zero vector is (0,0,0).

(d) The inverse vector of (a,b,c) is $-(a,b,c)$ or $(-a,-b,-c)$.

EXERCISE 7·8

1. Find the sum of $(3,5,9) + (4,-7,6)$.
   Find the sum of $(2,4,1) - 3(3,5,1) + 3(6,1,-7)$.

2. Does the associative property for addition hold in the set of vectors in a line? a plane? 3-D space?

3. Which of the following properties hold in the set of vectors in a line? in a plane? in a space?
   (a) closure under addition
   (b) commutative property for addition
   (c) multiplication by a scalar distributive over addition
   (d) closure under multiplication by a scalar
   (e) existence of an additive identity
   (f) existence of inverses.

4. Refer to the model you made in Project 1. Write an algebraic name for
   (a) the red vector
   (b) the blue vector
   (c) the yellow vector (two ways)

   Write algebraically the addition of the red and the blue vectors.

7·9 Magnitude of Vectors in 3-D

Refer to the model you made in Project 1 of Section 7·8. Carefully draw a line segment from the origin (0,0,0) to the point (3,4,0). Remember (3,4,0) is on the xy-plane. Let's find the magnitude of this vector, $\vec{OA}$.

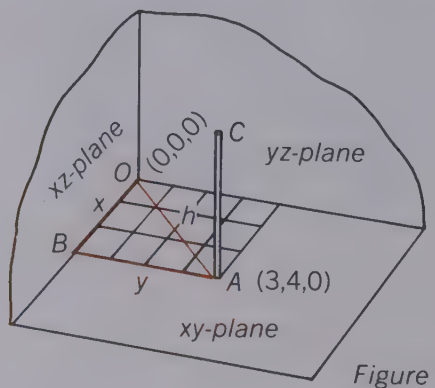


Figure 7-11

We note this vector is the hypotenuse of a vector triangle—a right triangle. Draw and color this triangle on your model. Hence we will use:

$$h^2 = x^2 + y^2$$
$$h^2 = 3^2 + 4^2$$
$$h = 5$$

The magnitude of $\vec{OA}$ is 5 units.

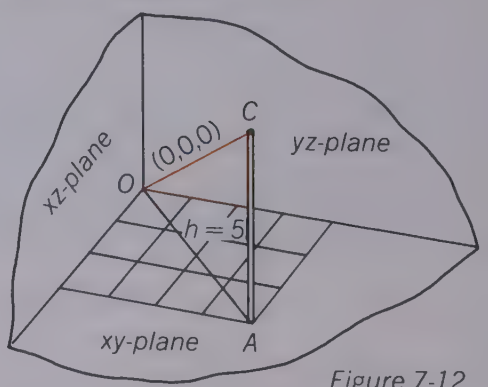


Figure 7-12

Now let us find the magnitude of the red vector, $\vec{OC}$. We note that the vector is the hypotenuse of the right triangle OAC. Cut from cardboard or transparent plastic, a right triangle to fit in your model. Color it appropriately. Tape into place.

In Figure 7-13:

$\vec{AC} = z = 5$ units

$\vec{OC} = H$ units

$H^2 = z^2 + h^2$

$= 5^2 + 5^2$

$H = 7.1$ units.

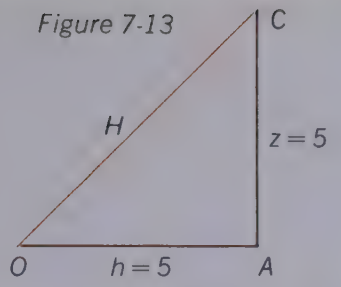
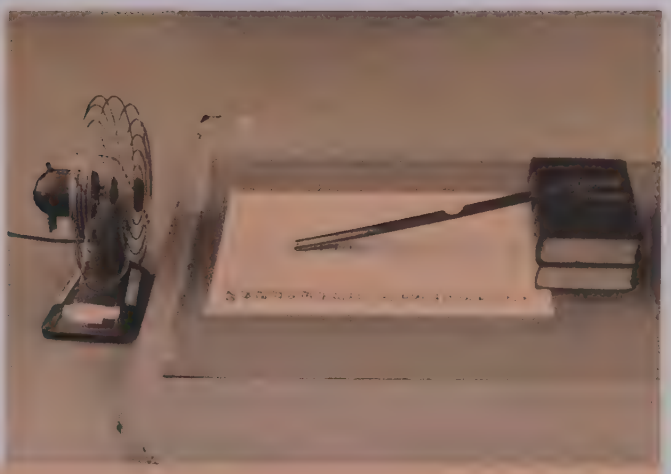


Figure 7-13

Therefore the magnitude of the vector, $\vec{OC}$, is 7.1 units. Check by using the ruler on your model.



Project

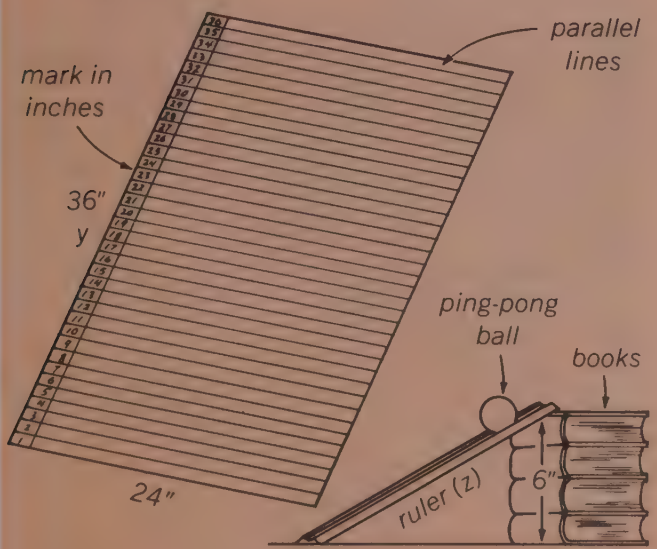
Project 1

Purpose

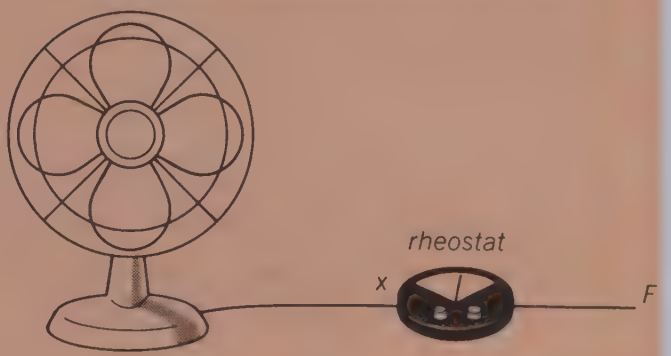
To illustrate a relation which can be represented in 3-D.

Method

1. Set up apparatus on floor or table top as shown. The paper with parallel lines 1 inch apart should be at least 24″ × 36″.



2. You will need to experiment with the position of the fan. If it is too close the ball will be blown completely off the paper when the current is high. If it is too far away the ball will roll straight down the lines when the current is low. The fan should be somewhere between these two extreme settings.



3. When the fan has been suitably placed, set the rheostat on the lowest position. Record the current (x). Release the ball from the 2″ mark on the ruler (z). Record the number on the y-scale when the ball crosses it (y).

4. Increase the rheostat by 1 position (this will now be $x = 2$). Release the ball from the 4″ mark. Record the y-value.

5. Repeat for 4 more successive rheostat settings, and with the ball at 6″, 8″, 10″, 12″ positions.

6. Write the relations as ordered pairs (x,y,z).

7. What do you think the y-value will be when $x = 7$? When $z = 14$?

8. Set up a 3-D model to represent the relation in step 7.

EXERCISE 7·9

1. Using the method just illustrated find the magnitude of the blue vector (10,9,13) in your model.

2. Calculate the magnitude of each vector. Draw representations of 3-D models of each as illustrated in Figures 7-11, 7-12, 7-13.

   (a) (1,1,1) (b) (5,12,4)

   (c) (4,8,9) (d) (3,7,6)

   (e) $(\sqrt{3},-2,-3)$ (f) $(\sqrt{5},\sqrt{7},\sqrt{6})$

3. An airplane has a forward speed of 600 miles per hour. There is a cross wind (at right angles to the course) of 200 miles per hour. An updraft of 300 miles per hour strikes the plane. Calculate the magnitude of the track of the plane.

4. Add:

   (a) (2,3,7) + (7,−7,8) (b) (7,−7,8) + (2,3,7)

   (c) (8,−3,−7) + (5,6,−9) (d) (5,6,−9) + (8,−3,−7)

   Do you think addition of vectors in 3-D is commutative? Give an example to illustrate your answer.

5. Add:

   (a) (7,−3,4) + [(5,6,−13) + (−7,−2,11)]

   (b) [(7,−3,4) + (5,6,−13)] + (−7,−2,11)

   (c) (11,0,3) + [(0,2,0) + (−3,−2,−4)]

   (d) [(11,0,3) + (0,2,0)] + (−3,−2,−4)

   From these four questions what generalized statement would you make about vectors in 3-D?

6. What is the additive inverse of (3,−7,13)?

7. Subtract:

   (a) (5,−6,−4) − (−7,−2,4)

   (b) (−7,−2,4) − (5,−6,−4)

   (c) (17,−21,23) − (14,−18,2)

   (d) (14,−18,2) − (17,−21,23)

   Is subtraction of vectors in 3-D commutative?

Using the Library

1. Add these names and events to your time line.

   Norman Conquest
   First Crusade
   Omar Khayyam – mathematician and poet
   Fibonacci – sequence
   Magna Carta
   Marco Polo – world traveller
   Dante
   Chaucer born
   Black Death rages through Europe
   Wycliffe's English Bible completed
   Uniform weights and measures established in England
   Joan of Arc burned at the stake
   Printing from a moveable type
   Vasco da Gama – set sail for India
   Dürer born – projective geometry and perspective in art
   Copernicus born – astronomer
   Columbus discovers America
   Leonardo da Vinci born – artist, mathematician, philosopher
   Michelangelo born – artist

2. Note several of the famous names in the latter part of this period. List several things for which Michelangelo and Leonardo da Vinci are known. Write a brief report on the historical significance of these two.

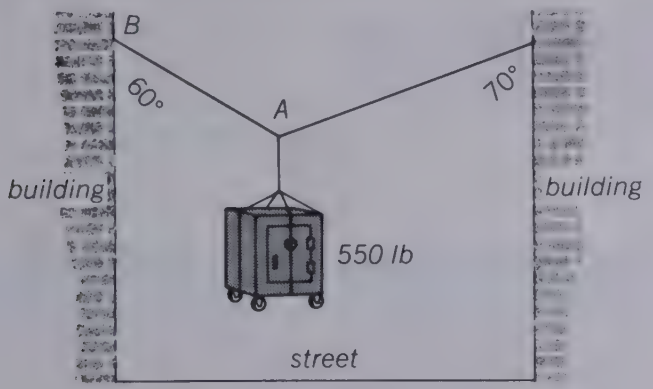
3. Albrecht Dürer was a mathematician and artist. Prepare a report referring to points that tie art and mathematics together. Dürer related the two. Obtain, if possible, a copy of Dürer's engraving "Melancholia". Endeavor to create in clay a reproduction of the polyhedra shown in the engraving. Make the polyhedra as simple as possible – that is with the least number of vertices, edges and faces. How many different shapes does your class get?

4. Research the aspect of projective geometry dealing with perspective in art. Explain by diagrams what *perspective* is. Show how perspective is related to art and used in some famous pictures.

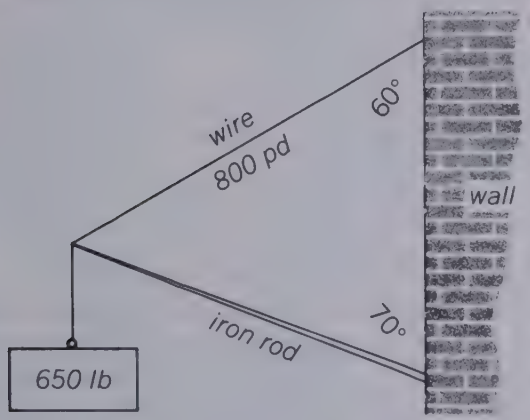
REVIEW EXERCISE 7

1. Use the Parallelogram Law of Vectors to add $(-1,-6) + (7,2)$.

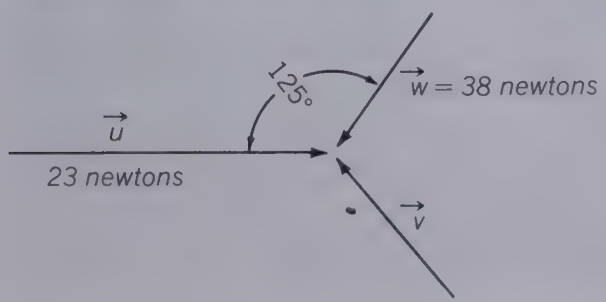
2. Find geometrically the forces acting in $\vec{AB}$.



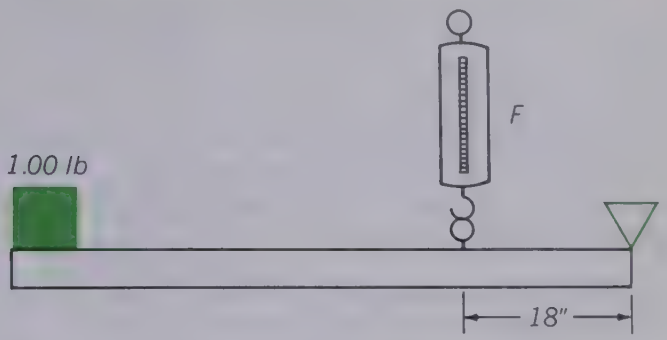
3. Calculate trigonometrically the force exerted by the iron rod.



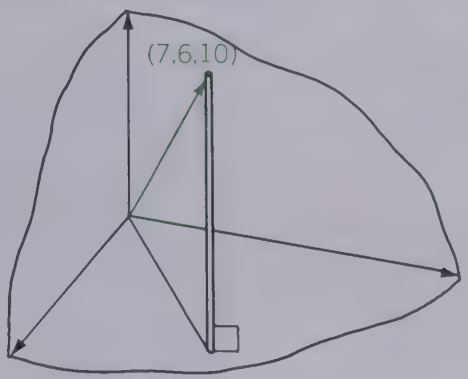
4. Calculate trigonometrically the magnetic force represented by $\vec{v}$ given the information in the diagram and that the system is in equilibrium.



5. Calculate the force F given that the lever is 10 feet long and weighs 16 pounds. Calculate the *M.A.*



6. Calculate the magnitude of the vector shown in the diagram.



7. A kite string is 60° to the horizontal. The tension in the string is 15 pounds. What can you determine about the force of the wind (horizontal) and the force of the lift (vertical)?

8. A forward horizontal velocity of a ball is 20 feet per second; its downward (vertical) velocity is 16 feet per second; a cross wind mutually at right angles to the two velocities, causes a velocity of 6 feet per second. Calculate the magnitude of the resultant track of the ball. Show the three vectors geometrically.

9. Find the single vector for each addition:

   (a) $3(2,-6,4) + 5(-3,4,11) - 8(7,-9,12)$

   (b) $9(14,-5,8) - 7(14,-5,8) + 12(14,-5,8)$

   (c) $12(3,\sqrt{3},\sqrt{7}) - 26(3,\sqrt{3},\sqrt{7}) + 8(3,\sqrt{3},\sqrt{7})$.

   Is there a short way to do parts (b) and (c)?

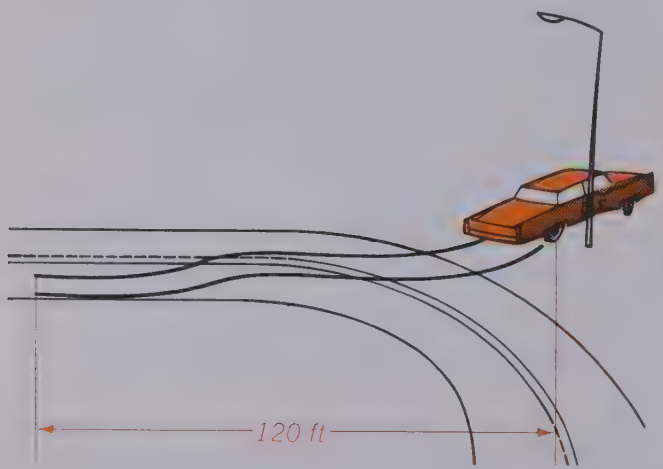
Your Job and Mathematics

Mathematics in the Police Department

Have you ever thought you would like to be a policeman or policewoman? Policemen work with people. You have to like people to be a policeman. Also you have to be able to do math.

1. Investigating automobile accidents is one of the duties of a police officer. Consider this case:

   A car on dry pavement on a city 30 mph zone leaves skid marks as shown.



   The policeman measures the length of the skid mark and applies this formula:

$$s = 5.6\sqrt{d \times f}$$

   where s is the speed in mph
   d is the length of skid marks in feet
   f is the resistance.

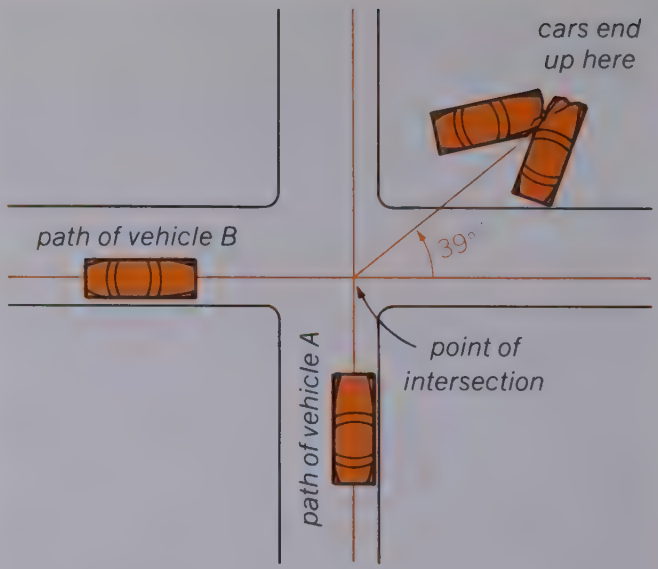
   The officer knows the resistance on dry pavement is 0.74.

   Should the officer charge the driver with speeding? Why?

2. An officer is investigating a two-vehicle collision at the intersection shown.

   Vehicle *A* weighs 1600 pounds.

   Vehicle *B* weighs 1000 pounds.



Driver of car *A* claims:

He was going 30 mph.
Driver of car *B* must have been going about 60 mph.

Driver of car *B* claims:

He was going about 30 mph.
Driver of car *A* was going 60 mph.

The officer measured the angle shown. He also calculated force of moving vehicle.

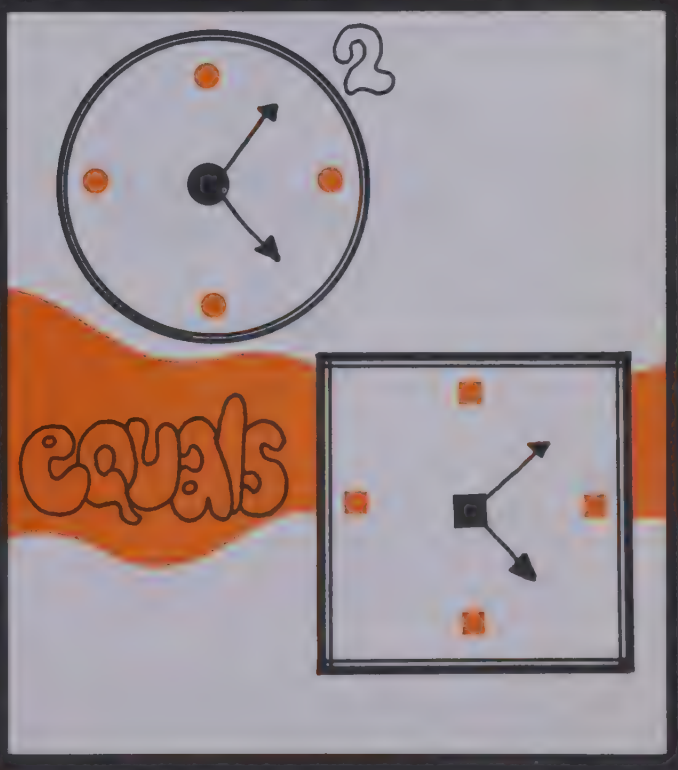
Who should the officer believe?

Who should the officer charge? Why? What other questions would you as the officer want answered (either by observation or asking questions) before laying a charge?

What information is missing?

Chapter Eight

POLYNOMIALS



8·1 Introduction: Product of Two Polynomials

In this chapter we will deal with the development of skills in the operations with polynomials.

> A polynomial is an expression of one term or the sum and/or difference of two or more terms.

Polynomials of one, two or three terms are called *monomials*, *binomials* or *trinomials*, respectively.

Examples of monomials (one term) are:

$$7, \quad x, \quad 3y, \quad 5abc$$

Examples of binomials (two terms) are:

$$2 - x, \quad 3a + 2b, \quad 6mn - 11$$

Examples of trinomials are:

$$x + y - 5, \quad x^2 + 4x + 4, \quad 3ab + 5bc - 2ac$$

The product of two polynomials can be illustrated as the area of a rectangle. Since the area of a rectangle is the product of length by the width, we can think of the length as one polynomial while the width is the other polynomial.

The following illustrations show the products of two polynomials as areas of rectangles.

(i)

	x	y
2	$2x$	$2y$

$2(x + y) = 2x + 2y$

(ii)

	$5a$	$2b$
$2a$	$10a^2$	$4ab$
b	$5ab$	$2b^2$

$$(2a + b)(5a + 2b)$$
$$= 10a^2 + 4ab + 5ab + 2b^2$$
$$= 10a^2 + 9ab + 2b^2$$

(iii)

	x	y	z
m	mx	my	mz
n	nx	ny	nz

$(m + n)(x + y + z) = mx + my + mz + nx + ny + nz$

Let us take illustration (ii) and follow it through without the aid of a rectangle.

$$\begin{array}{rl} 5a + 2b & \\ 2a + b & \\ \hline 10a^2 + 4ab & \textit{multiply } 5a + 2b \textit{ by } 2a \\ 5ab + 2b^2 & \textit{multiply } 5a + 2b \textit{ by } b \\ \hline 10a^2 + 9ab + 2b^2 & \textit{addition of like terms} \end{array}$$

or

$$\begin{aligned}(2a + b)(5a + 2b) &= 2a(5a + 2b) + b(5a + 2b) \\ &= 10a^2 + 4ab + 5ab + 2b^2 \\ &= 10a^2 + 9ab + 2b^2\end{aligned}$$

Example 1

Find the product of $5m - 3n + 4p$ by $7n - 2$.

Solution

(a)
$$\begin{array}{r} 5m - 3n + 4p \\ 7n - 2 \\ \hline 35mn - 21n^2 + 28np \\ -10m + 6n - 8p \\ \hline 35mn - 21n^2 + 28np - 10m + 6n - 8p \end{array}$$

or

(b)
$$\begin{aligned}&(7n - 2)(5m - 3n + 4p) \\ &= 7n(5m - 3n + 4p) - 2(5m - 3n + 4p) \\ &= 35mn - 21n^2 + 28np - 10m + 6n - 8p\end{aligned}$$

Example 2

Multiply $3xy - 5xz - 8yz$ by $4x + 3y$.

Solution

(a)
$$\begin{array}{r} 3xy - 5xz - 8yz \\ 4x + 3y \\ \hline 12x^2y - 20x^2z - 32xyz \\ -15xyz + 9xy^2 - 24y^2z \\ \hline 12x^2y - 20x^2z - 47xyz + 9xy^2 - 24y^2z \end{array}$$

or

(b)
$$\begin{aligned}&(4x + 3y)(3xy - 5xz - 8yz) \\ &= 4x(3xy - 5xz - 8yz) + 3y(3xy - 5xz - 8yz) \\ &= 12x^2y - 20x^2z - 32xyz + 9xy^2 - 15xyz - 24y^2z \\ &= 12x^2y - 20x^2z - 47xyz + 9xy^2 - 24y^2z\end{aligned}$$

Did you follow each step in this example?

EXERCISE 8·1

Express each product as a polynomial in its simplest form.

1. $a(x + y)$
2. $2a(3a - 4b)$
3. $3(2x + 7y - 4z)$
4. $-2m(15 + 3m - 4m^2)$
5. $5xy(3x + 4y)$
6. $(a + b)(a + 2b)$
7. $(p - 2r)(3p + 5r)$
8. $(6x - 5y)(x - 2y)$
9. $(5x - 3y)(x + 2xy - y)$
10. $(2ab + 1)(4a - 3ab + 2b)$
11. $(x^2 + 2xy)(y^2 - 3xy)$
12. $(a - 3b)(a^2 + 3ab + 9b^2)$
13. $(2m + 5n)(4m^2 - 10mn + 25n^2)$
14. $(3x - 4y)(9x^2 + 12xy + 16y^2)$
15. $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$
16. $(5 - 2x)^3$
17. $(3a^2 - ab + 2b^2)^2$
18. $(m - n)^2(m^2 + 2mn + n^2)$

8·2 Division of Two Polynomials

To divide one polynomial by another, a useful tool is the finding of a common factor of both polynomials. Thus, to divide $6ab$ by $4bc$ begin by expressing the operation in the form of a fraction. That is,

$$6ab \div 4bc = \frac{6ab}{4bc}.$$

The common factor of $6ab$ and $4bc$ is $2b$.

Therefore $$\frac{6ab}{4bc} = \frac{2b(3a)}{2b(2c)}.$$

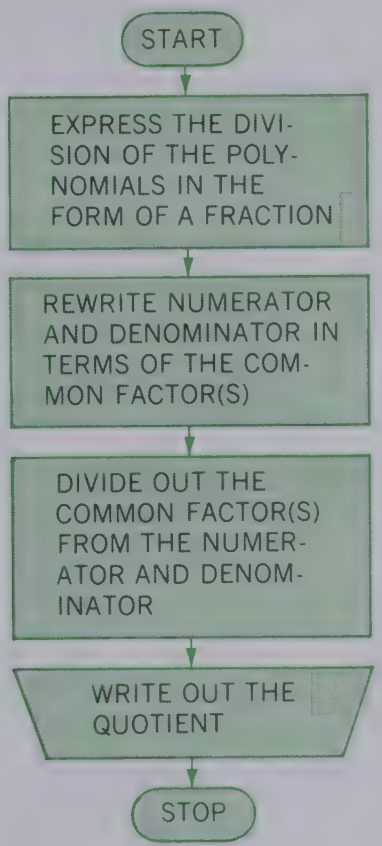
Then divide both numerator and denominator by the common factor $2b$, thereby reducing the fraction to

$$\frac{3a}{2c}.$$

The whole process then appears as follows:

$$6ab \div 4bc = \frac{6ab}{4bc} = \frac{2b(3a)}{2b(2c)} = \frac{3a}{2c}.$$

We can chart the division of two polynomials in this way.


START
EXPRESS THE DIVISION OF THE POLYNOMIALS IN THE FORM OF A FRACTION
REWRITE NUMERATOR AND DENOMINATOR IN TERMS OF THE COMMON FACTOR(S)
DIVIDE OUT THE COMMON FACTOR(S) FROM THE NUMERATOR AND DENOMINATOR
WRITE OUT THE QUOTIENT
STOP

Example 1

Divide $15x^2y - 21xy^2$ by $3xy$.

Solution

$$\frac{15x^2y - 21xy^2}{3xy} = \frac{3xy(5x - 7y)}{3xy} = 5x - 7y$$

Example 2

Divide $6a^2 + 5ab - 4b^2$ by $2a - b$.

Solution

$$\frac{6a^2 + 5ab - 4b^2}{2a - b} = \frac{(2a - b)(3a + 4b)}{(2a - b)} = 3a + 4b$$

or

$$\begin{array}{r} 3a + 4b \\ 2a - b \overline{)\,6a^2 + 5ab - 4b^2} \\ \underline{6a^2 - 3ab} \\ 8ab - 4b^2 \\ \underline{8ab - 4b^2} \\ 0 \end{array}$$

This method follows the procedure of long division.

Were you able to follow every step in this procedure?

Example 3

Calculate $(27x^3 + 8y^3) \div (3x + 2y)$.

Solution

Rewrite $27x^3 + 8y^3$ in descending order of x and ascending order of y. The reason becomes evident in the long division procedure.

$$\begin{array}{r} 9x^2 - 6xy + 4y^2 \\ 3x + 2y \overline{)\,27x^3 + 0x^2y + 0xy^2 + 8y^3} \\ \underline{27x^3 + 18x^2y} \\ -18x^2y + 0xy^2 \\ \underline{-18x^2y - 12xy^2} \\ 12xy^2 + 8y^3 \\ \underline{12xy^2 + 8y^3} \\ 0 \end{array}$$

EXERCISE 8·2

Express each fraction in its simplest form.

1. $\frac{18a^2b^2}{12ab}$
2. $\frac{9x^3y^2z}{6x^2y^2}$
3. $\frac{12x^2 - 15x}{3x}$
4. $\frac{16x^2y^2 + 12x^2y - 20xy^2}{4xy}$
5. $\frac{42m^3n^2p + 24m^2np^3 - 30mn^3p^2}{6mnp}$

Divide the following as indicated.

6. $(a^2 - 2a - 15) \div (a + 3)$
7. $(6a^2 - ab - 12b^2) \div (2a - 3b)$
8. $(8m^3 - n^3) \div (2m - n)$
9. $(27x^3 - 1) \div (3x - 1)$
10. $(64a^3 + 125b^3) \div (4a + 5b)$
11. $(16x^4 - y^4) \div (2x + y)$
12. $(6a^3 - 3a^2b + 4ab + 3a^2 - 4b^2 + 2b) \div (2a - b + 1)$
13. $(a^4 - 2a^2b^2 + b^4) \div (a^3 + a^2b - ab^2 - b^3)$
14. $(16x^4 - 72x^2y^2 + 81y^4) \div (4x^2 - 9y^2)$
15. $(2m^4 + 5m^3 - 17m^2 + 13m - 3) \div (2m^2 - 3m + 1)$
16. $(100a^3 - 3a - 13a^2) \div (25a + 3)$

8·3 Factoring Expressions of the Form $ax^2 + bx + c$

$$a, b, c \in I, a > 0$$

Let us consider graphical representations of expressions of the form $ax^2 + bx + c$.

The expression $x^2 + 8x + 15$ may be represented by

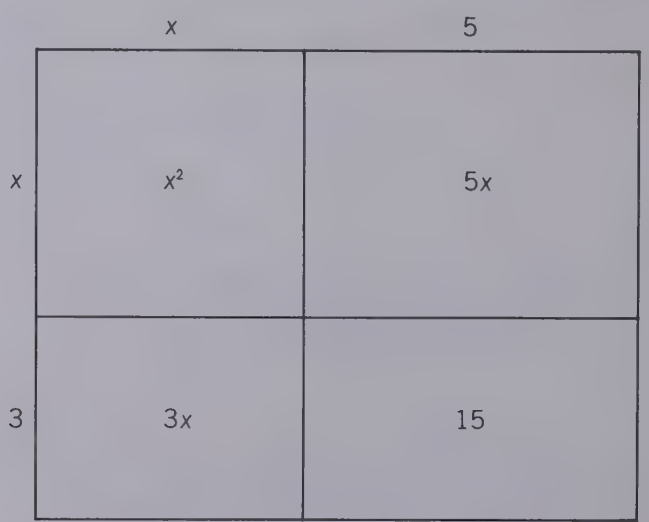


Figure 8-1

where $x^2 + 5x + 3x + 15 = x^2 + 8x + 15$.

The expression $2x^2 + 7x + 3$ may be represented by

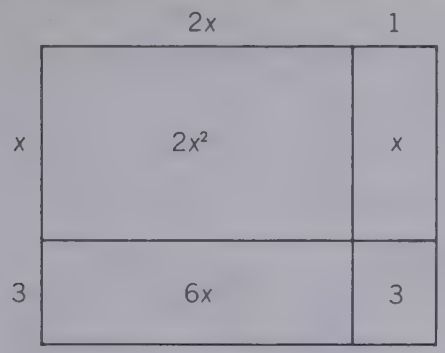


Figure 8-2

where $2x^2 + x + 6x + 3 = 2x^2 + 7x + 3$.

Similarly, the expression $x^2 + 3x - 10$ appears as

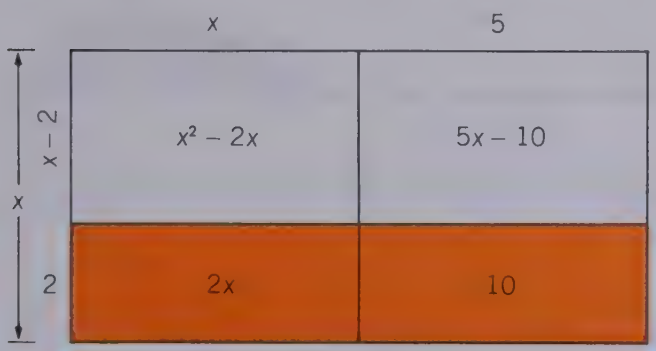


Figure 8-3

where $x^2 - 2x + 5x - 10 = x^2 + 3x - 10$

Note! The unshaded part is the actual representation of $x^2 + 3x - 10$.

To factor the expression

$$x^2 + 3x - 10$$

it will be of help to reason the above graphical representation in reverse. We observe that

$$x^2 + 3x - 10 = x^2 - 2x + 5x - 10.$$

You may wish to learn how to do this without resorting to a diagram.

The following procedure is suggested.

Step 1

Take the product of the first and last terms of the expression $x^2 + 3x - 10$. (i.e., x^2 and -10 yield a product of $-10x^2$)

Now find two factors of that product such that the sum of the factors equals the middle term (in this case find the factors of $-10x^2$ which yield the sum of $3x$).

$$x^2 + 3x - 10$$
$$-10x^2$$
$$(-2x)\ (5x).$$

Then, rewrite $x^2 + 3x - 10$
as $x^2 - 2x + 5x - 10.$

Step 2

Group the first two terms and the last two terms, so that expression appears like this:

$$(x^2 - 2x) + (5x - 10)$$

Step 3

Take out the common factor of each group.

$$x(x - 2) + 5(x - 2)$$

Step 4

Notice that $x - 2$ is the common factor of the whole expression. Take out $x - 2$ and the expression becomes

$$(x - 2)\ (x + 5)$$

which is the required factored form.

In step-by-step form the solution is

$x^2 + 3x - 10 = x^2 - 2x + 5x - 10$ Step 1
$= (x^2 - 2x) + (5x - 10)$ Step 2
$= x(x - 2) + 5(x - 2)$ Step 3
$= (x - 2)\ (x + 5)$ Step 4

$$x^2 - 2ax + a^2 = (x - a)^2$$

Example 1

Factor $2x^2 + 7x + 3$.

Solution

Refer to Figure 8-2.

$$2x^2 + 7x + 3$$
$$6x^2 \quad (6x)(x)$$

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + 6x + x + 3 && \text{Step 1} \\ &= (2x^2 + 6x) + (x + 3) && \text{Step 2} \\ &= 2x(x + 3) + 1(x + 3) && \text{Step 3} \\ &= (x + 3)(2x + 1) && \text{Step 4} \end{aligned}$$

Example 2

Factor $6x^2 + x - 35$.

Solution

$$6x^2 + x - 35$$
$$-210x^2 \quad (15x)(-14x)$$

$$\begin{aligned} 6x^2 + x - 35 &= 6x^2 + 15x - 14x - 35 && \text{Step 1} \\ &= (6x^2 + 15x) - (14x + 35) && \text{Step 2} \\ &= 3x(2x + 5) - 7(2x + 5) && \text{Step 3} \\ &= (2x + 5)(3x - 7) && \text{Step 4} \end{aligned}$$

EXERCISE 8·3

Factor:

1. $x^2 - 4x + 3$
2. $x^2 - x - 6$
3. $x^2 + 3x - 4$
4. $x^2 - 9x + 14$
5. $x^2 + 9x + 20$
6. $x^2 + x - 30$
7. $x^2 + 6x - 16$
8. $x^2 - 11x + 24$
9. $x^2 - 10x + 24$
10. $x^2 - 14x + 24$
11. Write a flow chart for factoring question 5.

Factor:

12. $x^2 - 18x + 81$
13. $a^2 + 16a + 64$
14. $2y^2 - 5y - 12$
15. $2b^2 - 9b + 9$
16. $2x^2 + x - 10$
17. $4x^2 - 12x + 9$
18. $3a^2 + 7a - 6$
19. $3x^2 + 13x + 12$
20. $4y^2 - 16y + 15$
21. $4x^2 - 8x - 21$
22. $5m^2 + 34m - 7$
23. $24z^2 - 67z + 8$
24. $42x^2 + x - 30$
25. $20a^2 - 73a + 63$
26. $28b^2 + 33b - 28$
27. $12y^2 - 40y - 7$
28. $27x^2 + 36x - 8$
29. $24x^2 - 55x + 25$
30. $35x^2 + 79x + 42$
31. $40x^2 - 3x - 54$

8·4 Rational Expressions

Algebraic fractions in which the numerators and denominators are polynomials are called *rational expressions*. Here are some examples of rational expressions:

$$\frac{x+1}{x-1}, \frac{4x-y}{x}, \frac{3x^3 - 24y^3}{x^2 + 2xy + 4y^2}$$

Since division by zero is not defined, any rational expression which has a variable in its denominator will not have meaning if the replacement for that variable results in a denominator equalling zero. Therefore, the rational expression

$$\frac{x+1}{x-1}$$

has no numerical value when

$$x - 1 = 0 \text{ or } x = 1.$$

Rational expressions lend themselves to the same mathematical operations that are performed on rational numbers. In most cases rational expressions are not in simplified form; hence, procedures in simplifying them are of great value to the solution of problems.

Here are some examples of these procedures with rational expressions.

Reduction of Rational Expressions

Example 1

Reduce to lowest terms: $\dfrac{x^2 + 3x - 10}{x - 2}$.

Solution

$$\frac{x^2 + 3x - 10}{x - 2} = \frac{(x - 2)(x + 5)}{x - 2} = x + 5$$

Example 2

Simplify: $\dfrac{x^2 + 3x - 4}{x^2 - 4x + 3}$.

Solution

$$\frac{x^2 + 3x - 4}{x^2 - 4x + 3} = \frac{\overset{1}{\cancel{(x-1)}}(x + 4)}{(x - 3)\underset{1}{\cancel{(x-1)}}} = \frac{x + 4}{x - 3}$$

Product of Rational Expressions

Example 3

Find the product in simplest form.

$$\frac{x^2 - x - 6}{2x^2 + 5x - 12} \times \frac{2x^2 - x - 3}{x^2 + 3x + 2}$$

Solution

$$\frac{x^2 - x - 6}{2x^2 + 5x - 12} \times \frac{2x^2 - x - 3}{x^2 + 3x + 2}$$

$$= \frac{(x - 3)\overset{1}{\cancel{(x+2)}}}{(x + 4)\underset{1}{\cancel{(2x-3)}}} \times \frac{\overset{1}{\cancel{(2x-3)}}\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+1)}}\underset{1}{\cancel{(x+2)}}}$$

$$= \frac{x - 3}{x + 4}$$

Division with Rational Expressions

Example 4

Divide and simplify:

$$\frac{3x^2 - 7x - 6}{2x^2 - x - 15} \div \frac{9x^2 - 9x - 10}{6x^2 + 5x - 25}$$

Solution

Factor each expression, invert and multiply.

$$\frac{\overset{1}{\cancel{(3x+2)}}\underset{1}{\cancel{(x-3)}}}{\underset{1}{\cancel{(2x+5)}}\underset{1}{\cancel{(x-3)}}} \times \frac{\overset{1}{\cancel{(3x-5)}}\overset{1}{\cancel{(2x+5)}}}{\underset{1}{\cancel{(3x+2)}}\underset{1}{\cancel{(3x-5)}}} = \frac{1}{1} = 1$$

EXERCISE 8·4

1. Reduce to lowest terms:

   (a) $\dfrac{m^2 - m - 12}{m^2 + 2m - 24}$ (b) $\dfrac{t^2 + t - 12}{t^2 + 4t - 21}$

   (c) $\dfrac{z^2 - 9z + 18}{3z^2 - 5z - 12}$ (d) $\dfrac{2a^2 + 3a - 20}{4a^2 - 16a + 15}$

   (e) $\dfrac{3x^2 + 20x - 7}{3x^2 + 8x - 3}$ (f) $\dfrac{3x^2 + 2x - 16}{6x^2 + x - 40}$

2. Simplify the following by performing the indicated operations.

   (a) $\dfrac{3a - 2}{2a + 5} \times \dfrac{2a^2 - a - 15}{3a^2 + 7a - 6}$

   (b) $\dfrac{x^2 - 2x - 3}{x^2 - x - 6} \times \dfrac{x^2 - 2x - 8}{x^2 - 5x + 4}$

   (c) $\dfrac{6a^2 + a - 1}{3a^2 + 2a - 8} \times \dfrac{3a - 4}{3a - 1}$ (d) $\dfrac{x - 5}{x + 3} \div \dfrac{x - 5}{x + 2}$

   (e) $\dfrac{2b^2 - 3b + 1}{2b^2 + 3b - 2} \div \dfrac{b^2 - 2b - 15}{b^2 - 3b - 10}$

3. Reduce to simplest form.

   (a) $\dfrac{10a^2 - a - 3}{20a^2 - 2a - 6}$ (b) $\dfrac{12x^2 + 4x - 21}{6x^2 + 13x + 6}$

   (c) $\dfrac{12b^2 + b - 20}{8b^2 - 34b + 30}$ (d) $\dfrac{12m^2 - 18m - 12}{18m^2 - 27m - 18}$

4. Simplify by performing indicated operations.

   (a) $\dfrac{3x^2 + 5x - 2}{2x^2 + 11x + 15} \times \dfrac{x + 3}{x + 2} \div \dfrac{3x - 1}{2x + 5}$

   (b) $\dfrac{5x^2 + 2x - 3}{4x^2 + 8x - 5} \times \dfrac{2x^2 + 7x + 5}{5x^2 - 8x + 3} \div \dfrac{x + 1}{2x - 1}$

   (c) $\dfrac{6b^2 + 23b + 20}{8b^2 + 18b - 5} \times \dfrac{15b^2 + 14b - 8}{8b^2 - 6b - 9} \div \dfrac{9b^2 + 24b + 16}{8b^2 - 14b + 3}$

8·5 Addition and Subtraction With Rational Expressions

To express the following as a single fraction:

$$\tfrac{1}{2}+\tfrac{2}{3}-\tfrac{3}{4}$$

we rewrite the fractions with equal denominators.

Thus,

$$\begin{aligned}\tfrac{1}{2}+\tfrac{2}{3}-\tfrac{3}{4} &= (\tfrac{1}{2}\times\tfrac{6}{6})+(\tfrac{2}{3}\times\tfrac{4}{4})-(\tfrac{3}{4}\times\tfrac{3}{3})\\ &= \tfrac{6}{12}+\tfrac{8}{12}-\tfrac{9}{12}\\ &= \tfrac{6+8-9}{12}\\ &= \tfrac{5}{12}\end{aligned}$$

Similarly, to express rational expressions like

$$\frac{4}{x}+\frac{3}{y}-\frac{2x+3y}{xy}$$

as a single fraction, we find the least common denominator. In this case it is xy. We then proceed as follows:

$$\begin{aligned}\frac{4}{x}+\frac{3}{y}-\frac{2x+3y}{xy} &= \left(\frac{4}{x}\times\frac{y}{y}\right)+\left(\frac{3}{y}\times\frac{x}{x}\right)-\frac{2x+3y}{xy}\\ &= \frac{4y+3x-(2x+3y)}{xy}\\ &= \frac{4y+3x-2x-3y}{xy}\\ &= \frac{x+y}{xy}\end{aligned}$$

Example 1

Add: $\dfrac{1}{x+y}+\dfrac{1}{x+2y}$.

Solution

$$\begin{aligned}\frac{1}{x+y}+\frac{1}{x+2y} &= \frac{x+2y}{(x+y)(x+2y)}+\frac{x+y}{(x+y)(x+2y)}\\ &= \frac{x+2y+x+y}{(x+y)(x+2y)}\\ &= \frac{2x+3y}{x^2+3xy+2y^2}\end{aligned}$$

Example 2

Subtract: $\dfrac{4}{x-y}-\dfrac{3}{x+y}$.

Solution

$$\begin{aligned}\frac{4}{x-y}-\frac{3}{x+y} &= \frac{4(x+y)-3(x-y)}{(x-y)(x+y)}\\ &= \frac{4x+4y-3x+3y}{x^2-y^2}\\ &= \frac{x+7y}{x^2-y^2}\end{aligned}$$

Before proceeding do questions 1 to 18, Exercise 8·5.

Example 3

Add $\dfrac{2a-1}{2a^2+3a-2}+\dfrac{3a+2}{3a^2+5a+2}+\dfrac{a}{a^2+3a+2}$.

Solution

First, factor all expressions:

$$\begin{aligned}&\frac{2a-1}{(2a-1)(a+2)}+\frac{3a+2}{(3a+2)(a+1)}+\frac{a}{(a+2)(a+1)}\\ &= \frac{1}{a+2}+\frac{1}{a+1}+\frac{a}{(a+2)(a+1)}\\ &= \frac{(a+1)+(a+2)+a}{(a+2)(a+1)}\\ &= \frac{3a+3}{(a+2)(a+1)}\\ &= \frac{3(a+1)}{(a+2)(a+1)}\\ &= \frac{3}{a+2}\end{aligned}$$

How might this student find mathematical ability useful?

EXERCISE 8·5

Perform the indicated operations.

1. $\frac{2}{a} - \frac{3}{a^2} + \frac{1}{a}$
2. $\frac{5}{a} + \frac{10}{b} - \frac{3a+7b}{ab}$
3. $\frac{x-4y}{xy} + \frac{2}{x} + \frac{1}{y}$
4. $\frac{5}{m} + \frac{2m-3n}{mn} + \frac{8}{n}$
5. $\frac{2}{r} - \frac{s+3r}{3rs} + \frac{2s-3r}{rs}$
6. $\frac{2a+3b}{ab} - \frac{3}{2b} + \frac{5}{3a}$
7. $\frac{3}{a+b} - \frac{4}{a-b}$
8. $\frac{6}{m+n} - \frac{6}{m-n}$
9. $\frac{2a}{a-b} + \frac{3b}{a+b}$
10. $\frac{4r}{2a+b} - \frac{5s}{3a-b}$
11. $\frac{3x}{x-2y} - \frac{3y}{x+2y}$
12. $\frac{4y}{2x-3y} - \frac{3}{2x+3y}$
13. $\frac{4x}{2x-y} - \frac{5x}{3x+2y}$
14. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$
15. $\frac{2a-b}{ab} - \frac{2a+b}{3ab}$
16. $\frac{3x-y}{6x} + \frac{2x-3y}{5y}$
17. $\frac{5}{x+y} + \frac{2}{x-y}$
18. $\frac{6}{a+2} - \frac{3}{a+3} + \frac{2a-5}{a^2+5a+6}$
19. $\frac{z-3}{z+1} + \frac{z-1}{z+3}$
20. $\frac{2b-5}{b+3} - \frac{b+4}{b-1} + \frac{3b^2+5b-7}{b^2+2b-3}$
21. $\frac{5x^2-2}{2x^2-x-28} - \frac{3x+1}{2x+7} - \frac{x+2}{x-4}$
22. $\frac{a+4}{2a+5} - \frac{a^2+9a+2}{6a^2+11a-10} - \frac{a-2}{3a-2}$
23. $\frac{a^2-2a-15}{a^2+a-2} - \frac{a^2+2a-3}{a^2+5a+6}$
24. $\frac{2x^2-9x+9}{x^2+x-12} + \frac{4x^2+4x-3}{2x^2+7x-4} - \frac{12x^2+5x-2}{3x^2+14x+8}$
25. $\frac{3}{b^2-b-2} - \left[\frac{3}{2b^2-b-6} - \frac{1}{2b^2+5b+3}\right]$

8·6 Solution of a Quadratic Equation

A general quadratic equation is of the form

$$ax^2 + bx + c = 0$$

where $a, b, c \in R$ and $a \neq 0$.

To solve a quadratic equation is to find the value or values for the variable which will satisfy the equation.

Three methods of solving a quadratic equation will be discussed in this section. The methods are by

(i) factoring,
(ii) completing the square, and
(iii) formula.

Factoring

If the product of two factors is zero, then either one factor or the other factor must equal zero. Of course, there is also the case where each factor is equal to zero.

Therefore, if $(x-3)(x+2) = 0$
then $x - 3 = 0$ or $x + 2 = 0$.

Solving each equation separately we get

$x = 3$ or $x = -2$.

The solution is then expressed

$x = \{-2, 3\}$.

Example 1

Solve for y: $y^2 - 3y - 28 = 0$.

Solution

By factoring $y^2 - 3y - 28 = 0$
we obtain $(y-7)(y+4) = 0$
then $y - 7 = 0$ or $y + 4 = 0$
and $y = 7$ or $y = -4$.

Solution set is $y = \{-4, 7\}$.

First substitute -4 for y in $y^2 - 3y - 28$ and simplify the expression.

Then substitute 7 for y.

What results do you obtain in each case?

What does this indicate?

Example 2

Solve for a: $3a^2 + 5a = 4a + 10$.

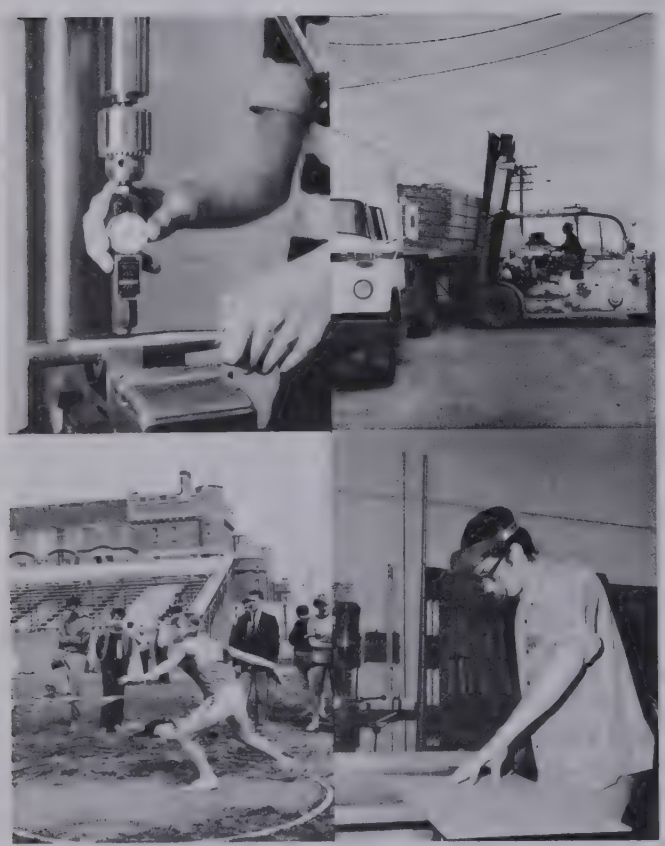
Solution

First, rewrite $3a^2 + 5a = 4a + 10$
in an equation of the form
$$ax^2 + bx + c = 0,$$
i.e., $3a^2 + a - 10 = 0.$

Then proceed to factor the left side.

$$3a^2 + 6a - 5a - 10 = 0$$
$$(3a^2 + 6a) - (5a + 10) = 0$$
$$3a(a + 2) - 5(a + 2) = 0$$
$$(a + 2)(3a - 5) = 0$$
$$a + 2 = 0 \text{ or } 3a - 5 = 0$$
$$a = -2 \text{ or } 3a = 5$$
$$a = \tfrac{5}{3}$$

Solution set is $a = \{-2, \tfrac{5}{3}\}$.



Discuss how mathematical equations might be involved with what the people in these photographs are doing.

EXERCISE 8·6

1. Write a flow chart for solving Example 2.

Solve the following.

2. $x^2 + 3x + 2 = 0$
3. $a^2 - 4a + 3 = 0$
4. $y^2 - y - 6 = 0$
5. $b^2 + 7b + 12 = 0$
6. $m^2 + 7m + 6 = 0$
7. $y^2 + 5y - 14 = 0$
8. $x^2 - 3x - 40 = 0$
9. $y^2 + 7y - 18 = 0$
10. $a^2 + 12a + 36 = 0$
11. $x^2 - 8x + 16 = 0$
12. $2b^2 + b - 3 = 0$
13. $2y^2 - 7y + 3 = 0$
14. $6y^2 + 5y - 6 = 0$
15. $8x^2 - 2x - 3 = 0$
16. $4x^2 + 12x + 5 = 0$
17. $6a^2 - 4a - 10 = 0$
18. $8a^2 + 6a - 9 = 0$
19. $3y^2 + 25y + 28 = 0$
20. $6m^2 - 31m + 18 = 0$
21. $4x^2 + 8x - 21 = 0$
22. $4x^2 - 7x = 5x - 9$
23. $16x^2 - 7x = 25 - 7x$
24. $5a^2 + 12a = 15a + 36$
25. $12a^2 + 7 = 44a$
26. $32m + 5m^2 - 20 = 40 - 10m^2$

Project

Project 1

Purpose

To make a slide rule to solve quadratic equations. We shall call it a *Quadratic Slide Cylinder* or QSC.

Method

1. Obtain a cardboard cylinder about 3″ in diameter and 10″ long.
2. Cut a sheet of graph paper to completely cover the lateral surface of the cylinder.
3. On the graph paper, draw a set of axes and mark on appropriate scale. A satisfactory range is $-8 \leq x \leq 8$ and $-25 \leq y < 25$.
4. Wrap the graph paper around the cylinder and glue or tape in place.
5. On a second sheet of graph paper and with the same scale as on the first sheet of graph paper, draw the graph of $y = x^2$.
6. Place a clear transparent sheet over the upper part of the grid and trace the curve and vertical axes. Extend the arms of the curve on the plastic.
7. Wrap the transparent sheet around the cylinder so that the two y-axes coincide and the graph has a minimum. Tape to form a sliding sleeve. This sleeve should be able to move horizontally and vertically.

How to Use

1. Reduce all quadratic equations to the form, $x^2 + bx + c = 0$.
2. Calculate the axis of symmetry: $\frac{-b}{2}$.
3. Turn sleeve until the y-axis coincides with the axis of symmetry on the graph paper.
4. Moving the sleeve vertically only, slide the sleeve until the curve intersects the y-axis on the graph paper at c.
5. The roots of the equation are obtained by identifying the points where the curve intersects the x-axis.
6. The solution for $x^2 + 4x - 12 = 0$ is shown to be $x = -6$ and $x = +2$.

Example

Change $4x^2 - 6x = 7$ to the $x^2 + bx + c = 0$ form:

$4x^2 - 6x = 7$

$4x^2 - 6x - 7 = 0$ *Add −7 to each side*

$x^2 + \frac{6}{4}x - \frac{7}{4} = 0$ *Divide each side by 4.*

$x^2 - \frac{3}{2}x - \frac{7}{4} = 0$ *Required form*

$b = \frac{-3}{2}$

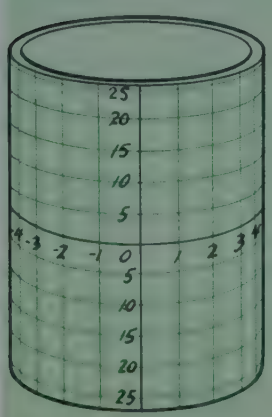
$c = \frac{-7}{4}$

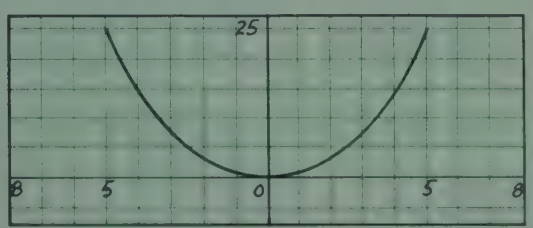
Questions

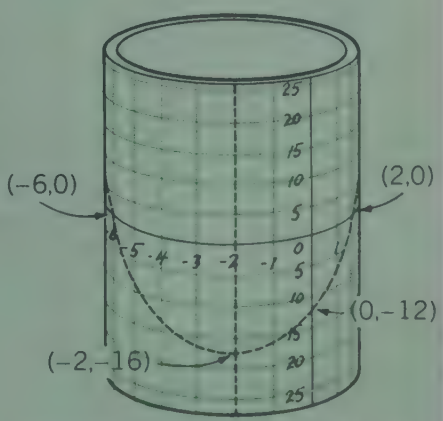
Solve using the QSC.

1. $x^2 + 3x - 4 = 0$
2. $x^2 - 2x - 3 = 0$
3. $x^2 - x - 20 = 0$
4. $4x^2 + 2x - 12 = 0$
5. $9x^2 - 27x = -8$
6. $4x^2 - 8x - 16 = 5$
7. $x^2 + x + 1 = 0$
8. $2x^2 + 4x - 15 = 15$

Could you solve them all? Why?







Solution for $x^2 + 4x - 12 = 0$: $\{-6,2\}$

8·7 Completing the Square

Example 1

Solve $x^2 + 6x + 8 = 0$
by the method of completing the square.

Solution

$$x^2 + 6x + 8 = 0$$

Complete square on left side:
$$x^2 + 6x = -8$$
$$x^2 + 6x + 9 = -8 + 9$$
$$x^2 + 6x + 9 = 1$$

(Take half of 6 and square it; i.e., 9)

Take square roots of both sides:
$$(x + 3)^2 = 1$$
$$x + 3 = \pm\sqrt{1}$$
$$x + 3 = \pm 1$$
$$x = -3 \pm 1$$

Therefore $x = -3 + 1$ or $x = -3 - 1$
$x = -2$ or $x = -4$

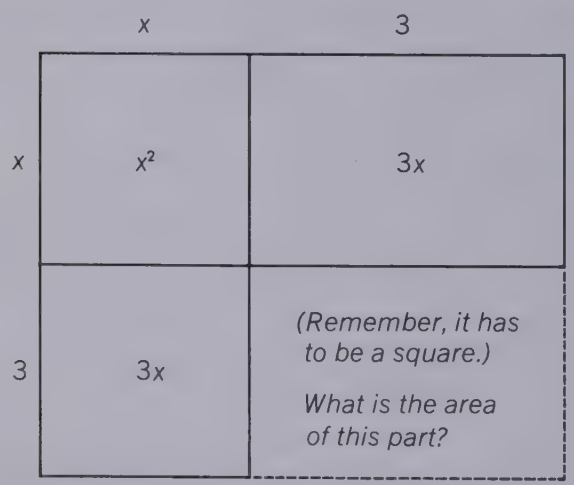
Solution set is $x = \{-2, -4\}$.

At this point we can show graphically how to complete the square. Let us begin with the line

$$x^2 + 6x = -8$$

from Example 1 above.

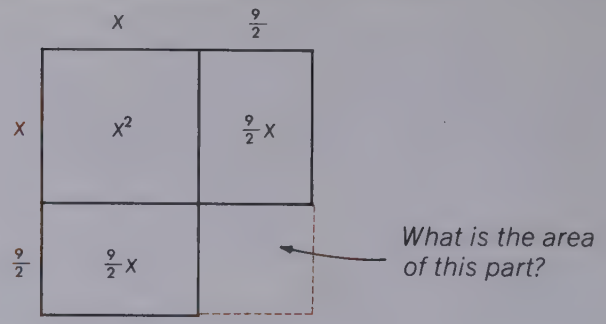
We can show the area for $x^2 + 6x$ in this illustration.



To complete the square we need $3^2 = 9$.

Hence, $x^2 + 6x + 9$ is the completed square.

Let us use the same type of illustration to show how to complete the square for $x^2 + 9x + \blacksquare$.



Notice each rectangle in the diagram – it is half of $9x$. To complete the square, a portion equal to $\frac{81}{4}$ must be added. Therefore, the completed square is

$$x^2 + 9x + \tfrac{81}{4}.$$

Example 2

By completing the square, solve

$$3x^2 + 7x - 20 = 0.$$

Solution

$$3x^2 + 7x - 20 = 0$$
$$3x^2 + 7x = 20$$
$$x^2 + \tfrac{7}{3}x = \tfrac{20}{3}$$

To complete the square the coefficient of x^2 must be 1. Then take $\frac{1}{2}$ of the coefficient of x and square it. Add this to both sides of the equation $(\frac{1}{2} \times \frac{7}{3})^2 = (\frac{7}{6})^2$.

$$x^2 + \tfrac{7}{3}x + \tfrac{49}{36} = \tfrac{20}{3} + \tfrac{49}{36}$$
$$(x + \tfrac{7}{6})^2 = \tfrac{289}{36}$$

(Take square roots of both sides)
$$x + \tfrac{7}{6} = \pm\tfrac{17}{6}$$
$$x = -\tfrac{7}{6} \pm \tfrac{17}{6}$$

Therefore, $x = -\frac{7}{6} + \frac{17}{6}$ or $x = -\frac{7}{6} - \frac{17}{6}$
$x = \frac{10}{6} = \frac{5}{3}$ or $x = -\frac{24}{6} = -4$

Solution set is $x = \{-4, \frac{5}{3}\}$.

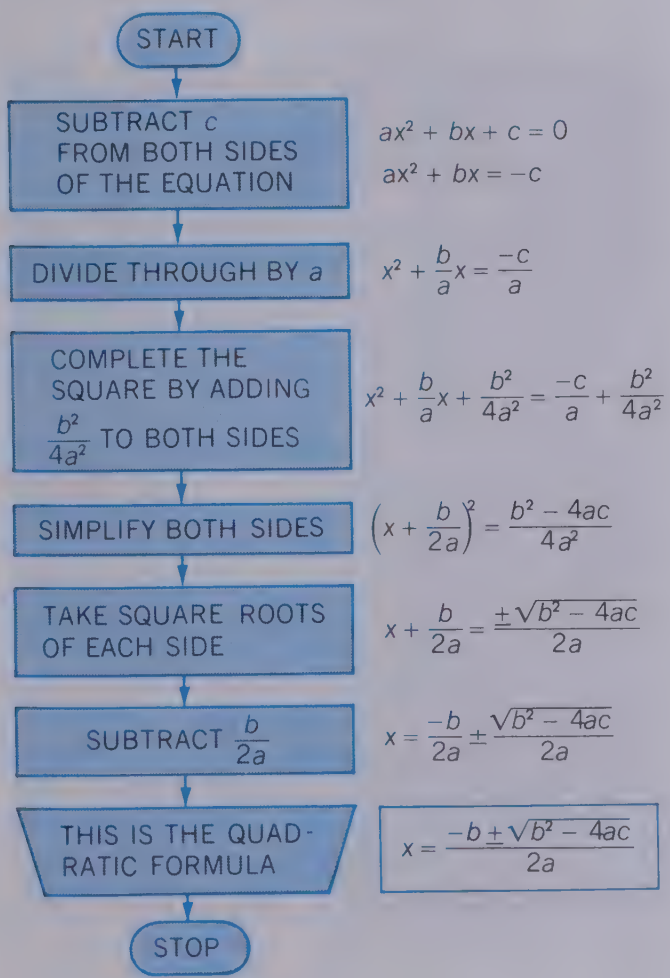
EXERCISE 8·7

1. Show graphically how to complete the square of each:
   (a) $x^2 + 8x$ (b) $x^2 + 10x$ (c) $x^2 + x$.
2. Write a flow chart for the solution to Example 2.

By completing the square, solve:

3. $x^2 + 4x + 3 = 0$
4. $x^2 - 6x + 8 = 0$
5. $a^2 + 2a - 15 = 0$
6. $m^2 + 4m - 21 = 0$
7. $t^2 - 10t + 24 = 0$
8. $y^2 - 10y + 21 = 0$
9. $x^2 - 3x - 10 = 0$
10. $b^2 + 5b - 6 = 0$
11. $r^2 + 9r + 18 = 0$
12. $2a^2 + 12a + 18 = 0$
13. $3a^2 - 9a + 6 = 0$
14. $4x^2 - 16x + 15 = 0$
15. $2a^2 - a - 21 = 0$
16. $3x^2 - 5x - 12 = 0$
17. $4y^2 + 17y - 15 = 0$
18. $6x^2 + x - 35 = 0$

8·8 Quadratic Formula


START
SUBTRACT c FROM BOTH SIDES OF THE EQUATION
$ax^2 + bx + c = 0$
$ax^2 + bx = -c$
DIVIDE THROUGH BY a
$x^2 + \frac{b}{a}x = \frac{-c}{a}$
COMPLETE THE SQUARE BY ADDING $\frac{b^2}{4a^2}$ TO BOTH SIDES
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$
SIMPLIFY BOTH SIDES
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
TAKE SQUARE ROOTS OF EACH SIDE
$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$
SUBTRACT $\frac{b}{2a}$
$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
THIS IS THE QUADRATIC FORMULA
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
STOP

Usually the general quadratic equation is

$$ax^2 + bx + c = 0 \text{ where } a, b, c \in I,\ a \neq 0.$$

We make use of this form to find a solution for x in terms of a, b, and c (See the flow chart).

The formula, then, is the solution of the general quadratic equation. There are two values for the variable x, i.e.,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These values of x we call the *roots* of the general quadratic equation.

Here we note that $b^2 - 4ac$ is under the radical sign. This, of course, can cause a certain problem if the value of $b^2 - 4ac$ should be negative.

Since there is no real number in the form of the square root of a negative value, the roots of the quadratic equation would then be nonreal or imaginary roots.

To apply the quadratic formula in the solution of a quadratic equation we must determine the values of a, b and c and substitute.

Thus, to solve

$$3x^2 + 7x - 6 = 0$$

we compare it with

$$ax^2 + bx + c = 0.$$

We note that $a = 3$, $b = 7$, and $c = -6$.

Substituting in $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

we get $x = \dfrac{-7 \pm \sqrt{7^2 - 4(3)(-6)}}{2(3)}$.

Simplifying, we get

$$x = \frac{-7 \pm \sqrt{49 + 72}}{6}$$

$$= \frac{-7 \pm \sqrt{121}}{6}$$

$$= \frac{-7 \pm 11}{6}$$

Then the roots of the equation are

$$x = \frac{-7 + 11}{6} \quad \text{or} \quad x = \frac{-7 - 11}{6}$$

$$x = \tfrac{4}{6} \quad \text{or} \quad x = -\tfrac{18}{6}$$

$$x = \tfrac{2}{3} \quad \text{or} \quad x = -3$$

Check to see that these, in fact, are the roots!

Example 1

By means of the quadratic formula find the roots of $5x^2 + 4x - 2 = 0$.

Solution

Identify a, b, and c.

$$a = 5, b = 4, c = -2$$

Substitute in the formula and simplify:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(5)(-2)}}{2(5)}$$

$$= \frac{-4 \pm \sqrt{56}}{10}$$

$$= \frac{-4 \pm 2\sqrt{14}}{10}$$

$$= \frac{-2 \pm \sqrt{14}}{5}$$

That is, the roots are

$$-\frac{2}{5} + \frac{\sqrt{14}}{5} \quad \text{and} \quad -\frac{2}{5} - \frac{\sqrt{14}}{5}.$$

Example 2

Solve $4y^2 - 7y + 5 = 0$.

Solution

$$a = 4, b = -7, c = 5$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(5)}}{2(4)}$$

$$= \frac{7 \pm \sqrt{49 - 80}}{8}$$

$$= \frac{7 \pm \sqrt{-31}}{8}$$

The roots are nonreal (imaginary) roots. They are imaginary since $\sqrt{-31}$ is not a real number. There is no real number a such that $a \times a = -31$ since $(-) \times (-) = +$ and $(+) \times (+) = +$.

The two imaginary roots are

$$\frac{7}{8} + \frac{\sqrt{-31}}{8} \text{ and } \frac{7}{8} - \frac{\sqrt{-31}}{8}.$$

EXERCISE 8·8

Solve by using the quadratic formula. Indicate whether the roots are real or nonreal.

1. $x^2 + 5x + 4 = 0$
2. $x^2 - x - 6 = 0$
3. $x^2 + 3x - 10 = 0$
4. $y^2 + 8y + 15 = 0$
5. $z^2 - 7z + 12 = 0$
6. $m^2 + 6m - 40 = 0$
7. $2x^2 + 5x - 12 = 0$
8. $6a^2 - 11a + 4 = 0$
9. $y^2 + 3y + 5 = 0$
10. $3z^2 - 6z - 2 = 0$
11. $x^2 - 6x + 9 = 0$
12. $9x^2 - 12x + 4 = 0$
13. $12b^2 + 3b + 1 = 0$
14. $4x^2 - 28x + 49 = 0$
15. $5x^2 + 5x - 12 = 0$
16. $7m^2 - 3m + 5 = 0$
17. $3a^2 + 12a - 2 = 0$
18. $8y^2 - 7y + 6 = 0$

Quadratic functions may be used to describe electrostatic forces.

8·9 Problem Solving

Certain problems, many of them related to life situations, can be solved by application of quadratic equations.

Let us examine two such examples.

Example 1

A number of students chartered a bus for \$80 to go skiing. At departure time 8 students did not show up. The students who went each had to pay 50 cents extra. How many students originally planned to go?

Solution 1

Let the original number of students be x.

Then, cost per student was to be $\frac{\$80}{x}$.

But, number of students who went was $x - 8$. Then, cost per student was $\frac{\$80}{x-8}$.

Since $\frac{\$80}{x-8}$ was more than $\frac{\$80}{x}$ by 50 cents we get the equation

$$\frac{\$80}{x-8} - \frac{\$80}{x} = \$.5$$

Multiply through by $x(x-8)$ and simplify to get the form of $ax^2 + bx + c = 0$. Then factor and solve for x.

$$\begin{aligned} 80(x) - 80(x-8) &= .5(x)(x-8) \\ 80x - 80x + 640 &= .5x^2 - 4x \\ .5x^2 - 4x - 640 &= 0 \\ x^2 - 8x - 1280 &= 0 \\ (x-40)(x+32) &= 0 \\ x - 40 = 0 \quad \text{or} \quad & x + 32 = 0 \\ x = 40 \text{ or } & x = -32 \end{aligned}$$

Since the number of students cannot equal -32, we discard it. Therefore, the answer is 40.

Solution 2

Let original number of students be x and cost per student be $\frac{\$80}{x}$.

We then know

(number of students) (cost per student) = Total cost.

We write this with variables and numbers.

$$(x \quad)\left(\frac{80}{x} \quad\right) = 80$$

But we know when number of students is decreased by 8 the cost per student is increased by $\frac{1}{2}$ dollar, therefore.

$$\begin{aligned} (x-8)\left(\frac{80}{x} + \frac{1}{2}\right) &= 80 \\ 80 + \frac{x}{2} - \frac{640}{x} - 4 &= 80 \\ \frac{x}{2} - \frac{640}{x} - 4 &= 0. \end{aligned}$$

Multiply through by $2x$ to get $ax^2 + bx + c = 0$ form.

$$\begin{aligned} x^2 - 1280 - 8x &= 0 \\ x^2 - 8x - 1280 &= 0 \end{aligned}$$

Factor: $(x-40)(x+32) = 0$

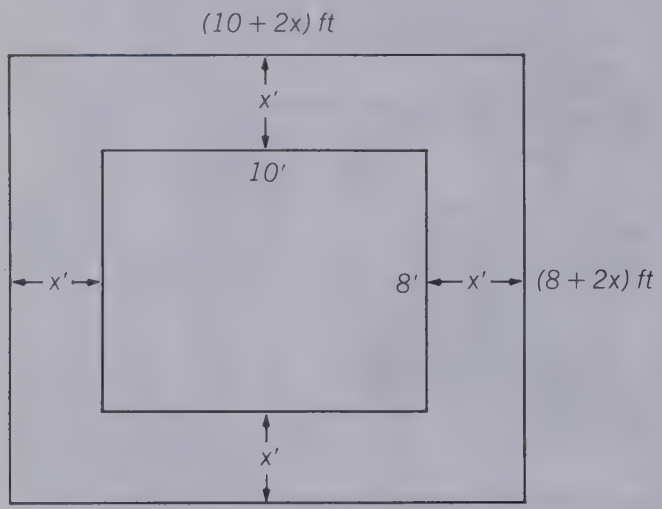
$$\begin{aligned} x - 40 = 0 \quad \text{or} \quad & x + 32 = 0 \\ x = 40 \text{ or } & x = -32 \end{aligned}$$

Solution set = $\{40\}$.

Example 2

A gardener wishes to enlarge an 8′ by 10′ flower plot so that it will be $2\frac{4}{5}$ times as large. He plans to do this by digging up an even width all around the original. How wide a strip must he dig?

Solution



Let width of strip be x feet.

The dimensions of the larger plot are
$10 + 2x$ feet by $8 + 2x$ feet.

Area of larger plot is

$$2\tfrac{4}{5}(10' \times 8') = 224 \text{ square feet.}$$

$$\therefore \quad (10 + 2x)(8 + 2x) = 224 \qquad \textit{Why?}$$

$$80 + 36x + 4x^2 = 224 \qquad \textit{Why?}$$

$$4x^2 + 36x - 144 = 0 \qquad \textit{Why?}$$

$$x^2 + 9x - 36 = 0 \qquad \textit{Why?}$$

$$(x + 12)(x - 3) = 0$$

$$x + 12 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -12 \quad \text{or} \quad x = 3$$

The width of the strip is 3 feet.

Why is $x = -12$ not a suitable answer?

EXERCISE 8·9

1. The volume of a rectangular box is 160 cubic inches. If the box is 4 inches high and the perimeter of its base is 26 inches, find its dimensions.

2. One side of a right triangle is x inches while the other side is $x + 2$ inches. If the hypotenuse is $x + 4$ inches find the sides of the triangle.

3. A rug is 9 feet long and 6 feet wide. By adding a border to the rug its area is increased by 34 square feet. What is the width of the border added?

4. Find two consecutive integers, the sum of whose squares is 85.

5. The area of a rectangular table top is 18 square feet. If the perimeter is 18 feet, find the dimensions of the table top.

6. A sports goods merchant paid \$150 for tennis racquets. He sold each racquet for \$5 more than he paid for it. When he sold all but two of the racquets, he had taken in \$10 more than he paid. How many racquets did he buy?

7. Two numbers differ by 2 and the sum of their squares is 202. Find the numbers.

8. A manufacturer produces two types of radios. One is marked \$4 more than the other. A wholesaler can buy 5 more radios for \$120 by ordering the cheaper type. What is the price of each type?

Using the Library

1. Continue to build your date line by adding name names and events. Write a brief comment about each mathematician.

   King Henry VIII becomes king of England
   Balboa discovers the Pacific
   Luther publishes his Ninety-Five Theses
   Magellan sails around the world
   Viète born
   Tycho Brahe born — his observations helped Kepler
   Napier born — logarithms
   Elizabeth becomes queen of England
   Shakespeare born
   Galileo born — musician, artist, mathematician
   Tartaglia
   Kepler born — three laws of planetary motion
   Galileo discovers law of pendulum
   England defeats Spanish Armada
   Descartes born — coordinate plane
   Desargues born

2. Describe the events leading to Descartes' death in 1650.

3. Niccolò Fontana received the name Tartaglia. How did he get this nickname? For what is Tartaglia noted in the mathematical world?

4. It is said that Cardano pirated from several mathematicians, yet Cardano did contribute to mathematics. What contributions did he make? Discuss the reported unconventionality of Cardano.

5. John Napier (Scottish) is noted for his discovery of logarithms. His uncle was Adam Bothwell, Bishop of Orkney, who assisted at the marriage of the Earl of Bothwell to Mary, Queen of Scots, and who also anointed and crowned the infant King James VI. What happened to Mary Queen of Scots? What were the circumstances that led to Mary's death?

6. Give details of why Galileo and Kepler were persecuted.

REVIEW EXERCISE 8

1. Expand the indicated product and collect like terms: $(3a + 2b)(2a^2 + ab - 3b^2)$.

2. Divide the following as indicated:
   $(8a^4 + 8a^3b - 27ab^3 - 27b^4) \div (2a^2 - ab - 3b^2)$.

3. Factor:
   (a) $a^2 - 6a - 16$ (b) $6x^2 + x - 1$
   (c) $2b^2 - 11b + 5$ (d) $12y^2 + 19y - 21$.

4. Simplify the following:
   (a) $\dfrac{x-3}{6} \times \dfrac{2x^2 - 4x + 2}{x^2 - 4x + 3}$
   (b) $\dfrac{y^2 - y - 2}{y^2 + y - 6} \times \dfrac{2y + 1}{y - 2} \div \dfrac{2y^2 + 3y + 1}{y^2 + 2y - 3}$
   (c) $\dfrac{2a - 5}{a + 3} + \dfrac{3a^2 + 5a - 7}{a^2 + 2a - 3} - \dfrac{a + 4}{a - 1}$.

5. Solve the following equations.
   (a) $2x^2 - x - 21 = 0$ (by factoring)
   (b) $2x^2 + 7x + 6 = 0$ (by completing the square)
   (c) $3x^2 + 5x - 12 = 0$ (by quadratic formula)

6. A table top is 21 square feet in area. If its perimeter is 19 feet, find the dimensions of the table.

Project

Project 1

Purpose

To explore employment opportunities in the motor transportation industry.

Method

Invite a manager of a large motor transport company serving your community to speak to your class. Ask him to speak about opportunities for both men and women in his company.

Prepare questions you will direct to him either in your letter or after he has spoken to you.

Suggestions:
1. opportunities
2. number employed
3. type of work
4. hours of work
5. wages
6. working conditions, etc.

MRC TRAINS DRIVERS FOR HEAVY TRUCKS

Between 1,000 and 2,000 new drivers are needed by Alberta's motor transportation industry every year.

That's a big bill to fill for a new course being sponsored by Mount Royal College.

The motor transportation course, in its first session, has already involved 31 students — 10 of whom have so far obtained permanent jobs in the trucking industry.

The students receive 12 weeks of training, six weeks of it classroom, the rest divided between on-the-job training and practice driving in everything from dump trucks to five-axle semis.

The trucks and on-the-job training are provided free by more than 10 local motor transportation companies. "It's a real example of co-operation between an industry and post-secondary education," says Tom Hale, MRC co-ordinator of program development.

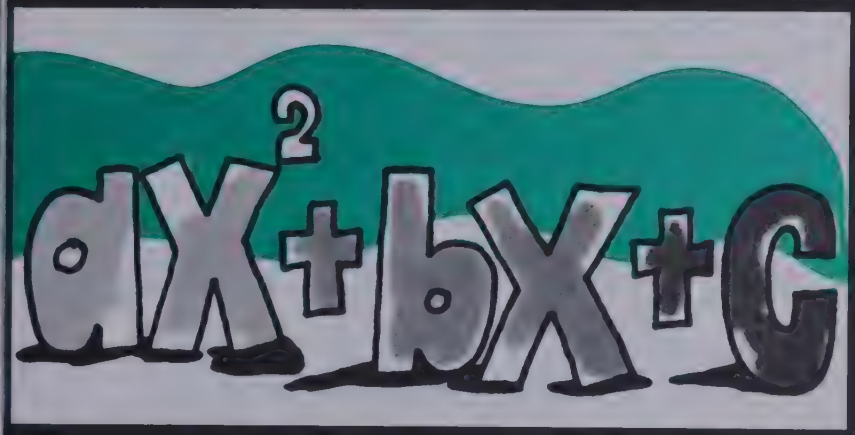
The course is funded jointly under the province's Priority Employment Training Program and Canada Manpower. The Alberta Motor Transportation Association has helped develop the curriculum and obtained loans of equipment and instructors from the industry.

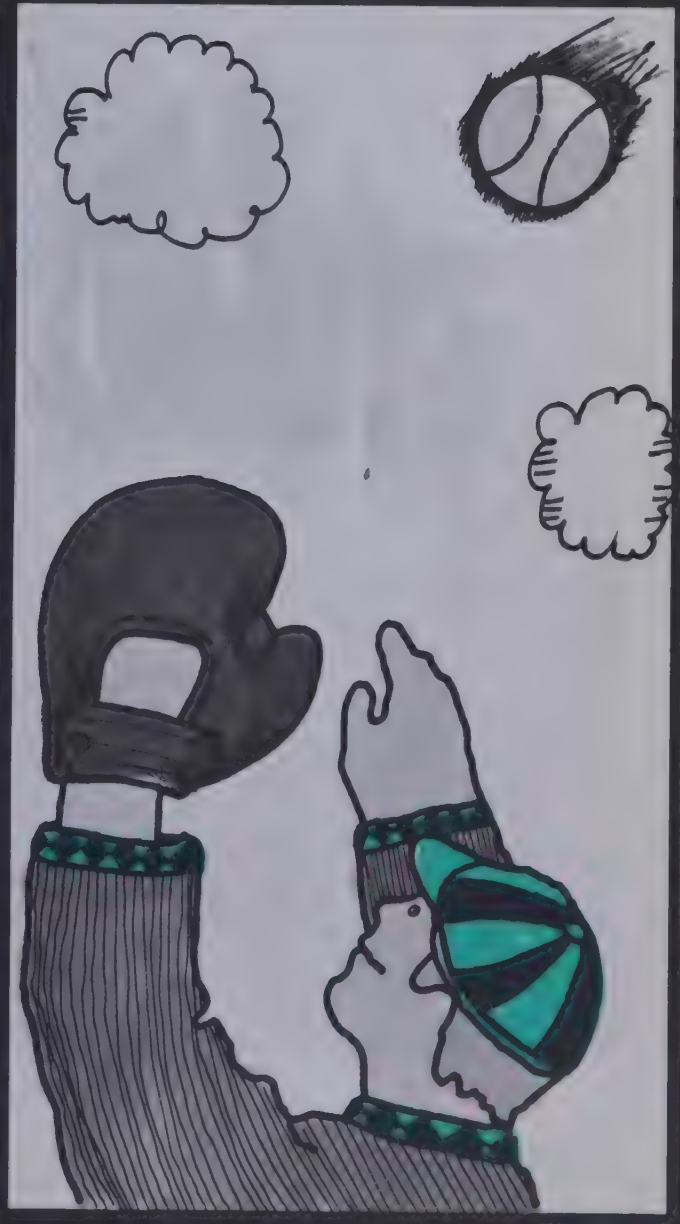
Plans are to continue the course on a regular basis at the MRC campus.

"Always circle check before you take off"...students are told in new trucking course.

Chapter Nine

QUADRATIC FUNCTIONS





9·1 Introduction

1. An object is projected vertically upwards with an initial velocity of 120 feet per second.

   (a) What is the maximum height reached by the object?

   (b) How long after projection before the object returns to ground level?

   $s = ut - 16t^2$

   s = height

   u = initial velocity

   t = time

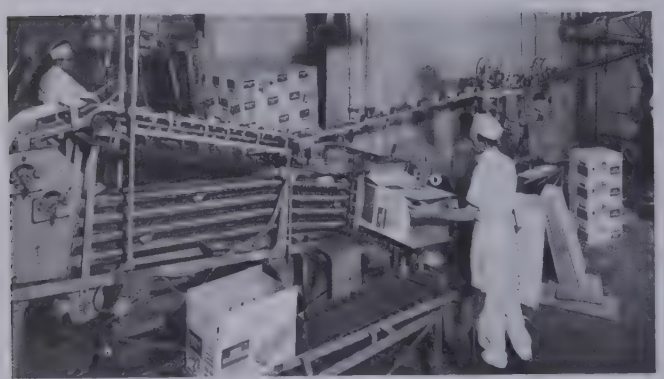
2. A farmer has 360 yards of fencing screen. He wishes to enclose a corral against an existing screen fence (i.e., he will use the screen for 3 sides only). What is the maximum area he can enclose with the fencing screen?

3. A manufacturer makes \$5.00 profit per ton of merchandise shipped from his warehouse when he has 100 tons on hand. Each week he delays shipment he can add 20 tons, but his profit is reduced 25¢ per ton.

   (a) How many weeks should he delay his shipment for maximum profit?

   (b) What is the maximum profit?

   (c) How many tons does he ship for maximum profit?



The above problems, and many others like them, appear in life situations frequently and require solutions. This type of problem falls under the study of Quadratic Functions.

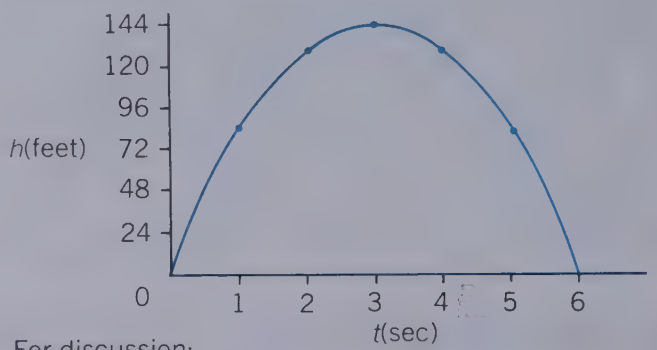
9·2 Graphing a Quadratic Function

The following will illustrate how we can use the quadratic function:

A batted ball rising vertically could follow the rule represented by $h = 96t - 16t^2$. This rule states the height (feet) is a function of time (seconds). To graph this function we first make a table of values: $h = 96t - 16t^2$.

t	0	1	2	3	4	5	6
h	0	80	128	144	128	80	0

Since all values are positive, the graph will be a first quadrant graph.



For discussion:

What is the maximum height reached by the batted ball?
How many seconds after it was batted did the ball reach maximum height?
How long was the ball in flight?
Does the graph of the function have symmetry?
What do we mean by symmetry?

EXERCISE 9·2

1. Make a table of values and graph the function $h = 48t - 6t^2$.

   (a) What is the maximum h of the graph?

   (b) For what t is h a maximum?

   (c) The graph is symmetrical about the line $t =$ ___.

2. Repeat question 1 for:
   (a) $h = 240t - 16t^2$
   (b) $h = 300t - 16t^2$
   (c) $h = 3200t - 16t^2$
   (d) $h = 64t - 8t^2$.
3. A rocket is put into vertical flight with a velocity of 2400 feet per second. How long does the rocket continue to rise? How high does it go? How long before it strikes the ground?

   Hint! Use the relation $h = ut - 16t^2$
   h = height in feet
   u = initial velocity in ft/sec
   t = time in seconds of flight

9·3 Investigations of Quadratic Functions

Note! It is most efficient to work in groups of three or more. Each student draws one or two graphs. The groups then compare all the graphs and discuss them in light of the questions. If the scale is the same for all graphs in each project the comparisons will be easy to make.

Projects

Project 1

Purpose

To investigate the graphs of $y = ax^2$.

Method

Let us begin with some of the simplest quadratic functions. They are:

$y = x^2$	$y = 2x^2$	$y = 3x^2$	$y = 5x^2$
$y = -x^2$	$y = -2x^2$	$y = -3x^2$	$y = -5x^2$

Make separate tables of values for each of the functions. Use the following as a guide:

for $y = \pm x^2$, $-6 \leq x \leq 6$
for $y = \pm 2x^2$, $-5 \leq x \leq 5$
for $y = \pm 3x^2$, $-5 \leq x \leq 5$
for $y = \pm 5x^2$, $-5 \leq x \leq 5$

Questions

1. Do the three graphs appear to have maximum or minimum y values? If so, what are these values?
2. As the coefficient of x^2 increases from 1 to 2 to 3, what is the effect on each graph?
3. Do the three graphs exhibit symmetry? If yes, about what line (called *axis of symmetry*) is each of them symmetrical?
4. What is the effect on the graphs if the coefficients of x^2 are negative (i.e., for $y = -x^2$, $y = -2x^2$ and $y = -3x^2$)?
5. Summarize what you have learned. Discuss your conclusions.

Project 2

Purpose

To investigate graphs of $y = ax^2 + c$.

Method

Draw graphs of:

(a)	$y = 2x^2 + 6$	$y = 2x^2 + 1$	$y = 2x^2 - 7$
(b)	$y = x^2 + 3$	$y = x^2 + 1$	$y = x^2 - 7$.

Again, make separate tables of values for each. Include sufficient values of the domain so as to make comparison possible.

Questions

1. Do the three graphs of each set have the same minimum values?
2. Do the three graphs of each set have the same axis of symmetry?
3. Are the minimum values found on the axis of symmetry?
4. What is the y-intercept, x-intercept of each graph?
5. What are the minimum values of the graphs?

6. What affect does the constant c have on the graph of ax^2?

7. In each of the functions, what part of $y^2 = ax^2 + c$ appears to affect or determine the minimum value?

8. Summarize what you have learned. Make a rule for each conclusion. Discuss your conclusions.

Project 3 (Optional)

Purpose

To investigate graphs of $x = ay^2$.

Method

Draw graphs of: $x = y^2$, $x = 2y^2$, $x = 3y^2$.

Questions

1. What are the coordinates of the maximum or minimum value of each relation? What do we mean by maximum or minimum in these cases?

2. As the coefficient of y^2 increases what is the effect on the graph?

3. What is the axis of symmetry for each?

4. Draw the graphs of: $x = -y^2$, $x = -2y^2$, $x = -3y^2$.

5. Answer questions 1, 2, and 3 for the graphs of question 4.

6. Summarize what you have learned. Discuss your conclusions.

Project 4 (Optional)

Purpose

To investigate graphs of $x = ay^2 + c$.

Method

Repeat Project 2 but draw graphs of:

(a) $x = 2y^2 + 6$ $x = 2y^2 + 1$ $x = 2y^2 - 7$

(b) $x = y^2 + 3$ $x = y^2 + 1$ $x = y^2 - 7$.

Questions

1. Do the three graphs have the same minimum values?

2. What are the coordinates of the minimum point on each graph?

3. What is the axis of symmetry of each graph?

4. What part of $x = ay^2 + c$ determines the minimum value of the relation?

5. What parts of $x = ay^2 + c$ determine the axis of symmetry?

6. Make a rule for each generalization you can draw from the above questions. Discuss your conclusions.

Project 5

Purpose

To investigate graphs of $y = (x + n)^2$.

Method

Draw graphs of:

$y = (x + 2)^2$ $y = -(x + 3)^2$ $y = (x + 5)^2$

$y = (x - 1)^2$ $y = (x - 3)^2$ $y = -(x - 4)^2$.

Questions

1. Write each equation in the form $y = ax^2 + bx + c$.

2. What is the axis of symmetry of each? Do you see a relation between the axis of symmetry and its function in question 1?

3. Compare each of these graphs with the graph of $y = x^2$ drawn in Project 1. What do you observe? How has the graph of $y = x^2$ been moved in order to get each of the 6 graphs?

4. Express in your own words how these six graphs differ from the graph of $y = x^2$.

5. What is the y-intercept of each? What relation do you see between each y-intercept and the respective function?

6. What are the coordinates of the minimum or maximum value of each graph?

7. What is (are) the x-intercept(s)?
   Without graphing, find for the graph of each function the:
   (a) axis of symmetry
   (b) coordinates of the maximum or minimum
   (c) x-intercepts
   (d) y-intercepts.

8. *Remember!* If you want to calculate the y-intercept, set $x = 0$, then solve for y. For example, in $y = (x + 2)^2$ set $x = 0$ and solve for y.
   $y = (0 + 2)^2$
   $y = 4$

9. Copy and complete these statements:
   To find the x-intercept,_____.
   In $y = (x + 2)^2$ set_____ and solve for____.

10. Calculate algebraically the x- and y-intercepts in 3 of the above equations. Do your answers check with your graphs?

11. The x-intercepts are called the _____ of the equation when $y = 0$. Another term for the x-intercepts when $y = 0$ is the *zeros* of the function.

12. Discuss your conclusions of the Project.

13. Optional: Investigate the graphs of $x = (y + m)^2$.

Project 6

Purpose

To investigate graphs of $y = a(x + n)^2 + c$.

Method

Draw graphs of:

(a) $y = (x + 1)^2 + 1$ (b) $y = (x + 1)^2 + 2$
(c) $y = (x + 2)^2 + 4$ (d) $y = 3(x + 1)^2 + 1$
(e) $y = 3(x + 1)^2 + 2$ (f) $y = 3(x - 2)^2 - 4$
(g) $y = -2(x + 1)^2 + 1$ (h) $y = -2(x + 1)^2 + 2$
(i) $y = -2(x - 2)^2 - 4$.

Label as many points as you can.

Questions

1. Write each equation in the form $y = ax^2 + bx + c$.

2. What is the axis of symmetry of each? Compare the functions in question 1 with the axes of symmetry.

3. What is the y-intercept of each?

4. Which of the graphs have maximums? minimums?

5. What is the minimum or maximum value in each? Compare your answers with the functions in question 1. What do you observe?

6. Which of the graphs cross the x-axis? What are the x-coordinates where the graphs cross the x-axis?

   Let us calculate the x-values when $y = 0$ (where the graphs cross the x-axis) for (f).

$$y = 3(x - 2)^2 - 4$$
$$0 = 3(x - 2)^2 - 4 \qquad y = 0$$
$$4 = 3(x - 2)^2 \qquad \textit{Why?}$$
$$\frac{4}{3} = (x - 2)^2 \qquad \textit{Why?}$$
$$\pm\sqrt{\frac{4}{3}} = x - 2 \qquad \textit{Why?}$$
$$2 \pm \sqrt{\frac{4}{3}} = x \qquad \textit{Why?}$$
$$2 \pm \frac{2}{3}\sqrt{3} = x \qquad \textit{Can you rationalize the denominator?}$$
$$x \simeq 3.1 \text{ or } x \simeq .9 \qquad \textit{Does this check with your graph?}$$

7. Use the method in question 6 to calculate the x-values for the other functions.

8. Without graphing find for the graph of each function, the:

   (a) axis of symmetry

   (b) maximum or minimum

   (c) x-intercepts (if necessary refer to Chapter 8.)

   Remember! $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

   (d) y-intercepts.

   $y = 4(x - 3)^2 - 9$ $\quad$ $y = 9(x + 4)^2 - 4$

   $y = 16(x - 8)^2 - 4$ $\quad$ $y = -25(x - 12)^2 + 16$

   $y = -6(x - 3)^2 + 5$ $\quad$ $y = 8(x + 1)^2 - 6$

9. The x-intercepts are called the _____ or _____ of the functions, when $y = 0$.

10. Discuss your conclusions of this Project.

11. Optional: Investigate the graphs of $x = a(y + m)^2 + c$.

Post Script

For those who need extra help:
The axis of symmetry for the graph of the function $y = ax^2 + bx + c$ is $x = \dfrac{-b}{2a}$.

The y-intercept is c.

The x-intercepts are the values of x for which $0 = ax^2 + bx + c$ is true. The x-intercepts are the roots or the zeros of the function.

The maximum value can be found by replacing x with the value of $\dfrac{-b}{2a}$ in $y = ax^2 + bx + c$.

Complete this:

$$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$

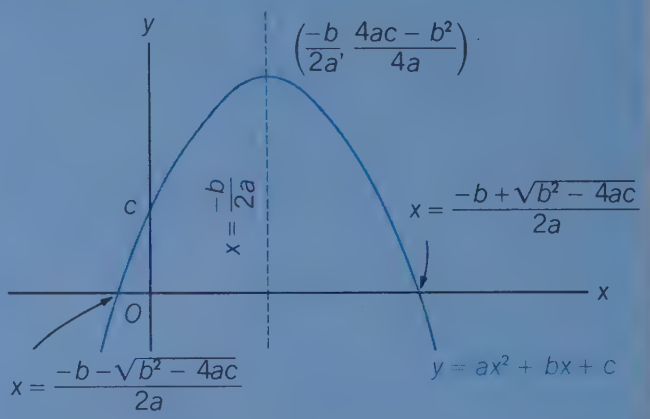
$$= a\left(\frac{\quad}{4a^2}\right) + \frac{\quad}{2a} + c$$

$$= \frac{\quad}{4} + \frac{\quad}{4a} + \frac{\quad}{4a}$$

$$= \frac{\quad}{4a}$$

$$y = \frac{4ac - b^2}{4a}$$

This is the y-value of the vertex of the graph. What is the x-value?



The coordinates of the vertex are:

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Project 7

Purpose

To investigate the graphs of $y = ax^2 + bx + c$.

Method

Graph the following:

$y = x^2 + 4x - 5$ $\quad$ $y = x^2 + 6x + 8$ $\quad$ $y = x^2 + 16x + 16$

$y = x^2 - 8x + 15$ $\quad$ $y = x^2 - 10x + 25$ $\quad$ $y = x^2 - 16x + 16$.

Hint! Change the function to an equivalent form – the form we used in Project 6 – $y = a(x + n)^2 + c$. Do you remember how to complete the square? See Section 8·6. We complete the square a little differently this time! Watch closely!

$y = x^2 + 4x + 5$

$= (x^2 + 4x + \_\_) + 5 - \_\_$ $\quad$ *Group. Why the negative sign?*

$= [x^2 + 4x + (\frac{4}{2})^2] + 5 - (\frac{4}{2})^2$ $\quad$ *Why* $(\frac{4}{2})^2$ *?*

$= (x^2 + 2)^2 + 5 - 4$ $\quad$ *Why?*

$y = (x + 2)^2 + 1$

Now refer to Project 6. Remember how to graph this now?

Continue. Then answer the questions.

After you have drawn the 6 graphs of the functions above, follow this example, then graph the following functions.

$y = 2x^2 - 8x + 6$

Change this to the form $y = a(x + n)^2 + c$.

$y = (2x^2 - 8x + \_\_) + 6 - \_\_$ *Group*

$= 2(x^2 - 4x + \_\_) + 6 - \_\_$ *Factor out 2.*

$= 2(x^2 - 4x + (\frac{4}{2})^2) + 6 - 2(\frac{4}{2})^2$ *Why 2 times?*

$= 2(x^2 - 4x + 2^2) + 6 - 8$ *Did you follow?*

$= 2(x - 2)^2 - 2$

Now graph this.

Graph the following after changing each to the form $y = a(x + n)^2 + c$.

(a) $y = 2x^2 + 8x + 6$ (b) $y = 2x^2 + 8x + 8$

(c) $y = 2x^2 + 12x + 18$ (d) $y = 3x^2 + 18x + 20$

Questions

1. What is the axis of symmetry in each graph?

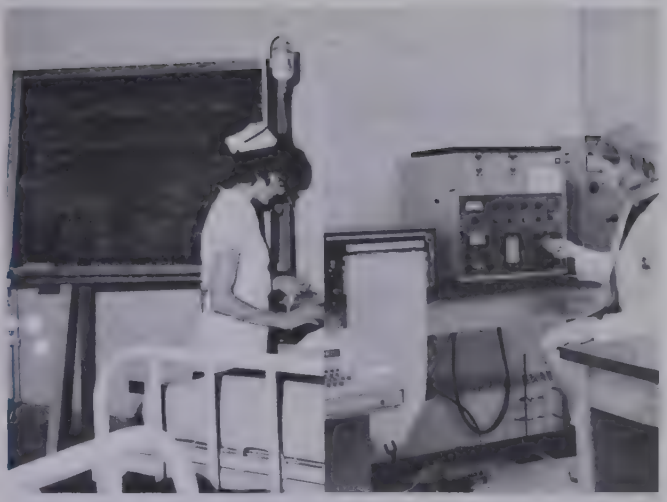
2. Sometimes $x = \frac{-b}{2a}$ is used to find the axis of symmetry of the graph of any function in the form $y = ax^2 + bx + c$. Check algebraically the axis of symmetry of the above graphs by using $x = \frac{-b}{2a}$.

3. Calculate the y-intercept for each graph by setting $x = 0$.

4. Calculate the x-intercepts for each by setting $y = 0$.

5. What are the coordinates of the vertex?

$ax^2 + bx + c$



Ease in dealing with numbers and equations is a great asset to people in all walks of life.

9·4 Using What We Know

Let us review what we know about graphing the function $y = ax^2 + bx + c$:

(a) the axis of symmetry is $x = \frac{-b}{2a}$,

(b) the maximum (or minimum) value is $y = \frac{4ac - b^2}{4a}$,

(c) the y-intercept is c,

(d) the x-intercepts are the zeros of the function (if real). These are found by setting $y = 0$ and solving for x in $0 = ax^2 + bx + c$. If possible factor and solve for x. If necessary we can always solve for x by using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1

A manufacturing company found that its profit (y) per unit on a special piece of equipment was related to its selling price (x) by the relation:

$$y = -4x^2 + 4x + 15$$

(a) What is the selling price that yields the maximum profit?

(b) What is the maximum profit?

Solution

We can make a table of values and draw the graph. Another way is to use the four items above and sketch the graph.

$$a = -4,\ b = 4,\ c = 15$$

(a) axis of symmetry:

$$x = \frac{-b}{2a}$$

$$x = \frac{-4}{2(-4)}$$

$$x = +\frac{1}{2}$$

(b) maximum:

$$y = \frac{4ac - b^2}{4a}$$

$$y = \frac{4(-4)(15) - (4)^2}{4(-4)}$$

$$= 16$$

(c) y-intercept is 15.

(d) We don't need to find the x-intercepts in this problem. But to sketch the graph a little more accurately we shall let $y = 0$.

Then: $-4x^2 + 4x + 15 = 0$

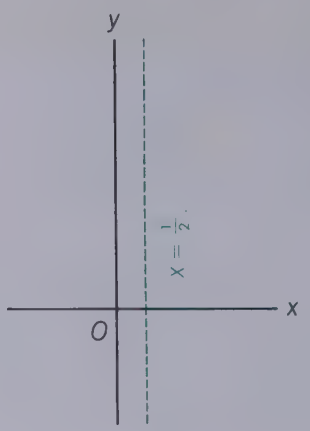
Factor $(2x + 3)(-2x + 5) = 0$

Find the zeros:

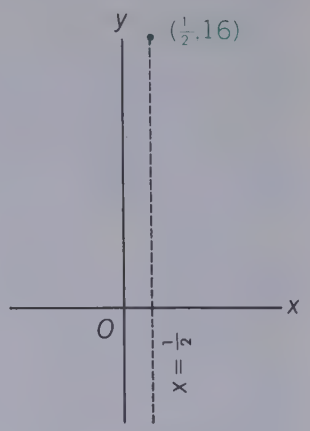
$2x + 3 = 0$ or $-2x + 5 = 0$

$x = -\frac{3}{2}$ $\qquad$ $x = \frac{5}{2}$

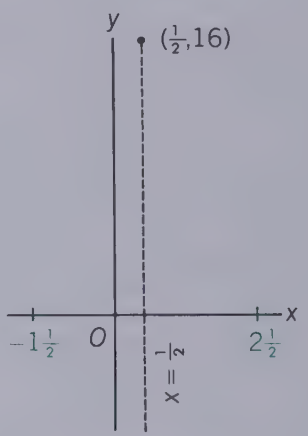
See how easy it is to sketch this graph now.



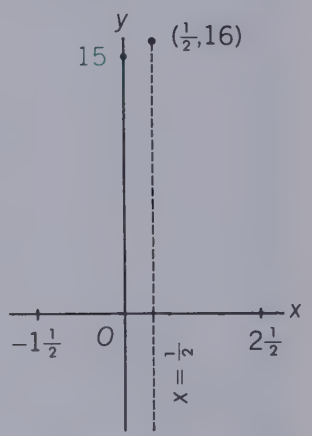
1. Draw axis of symmetry



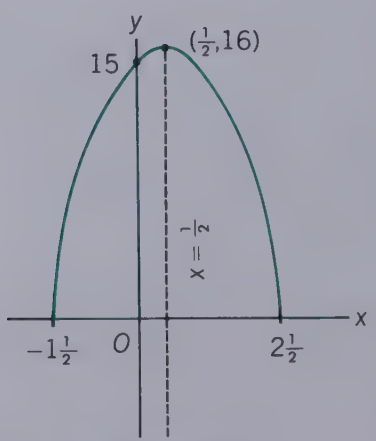
2. Mark vertex



3. Mark x-intercepts



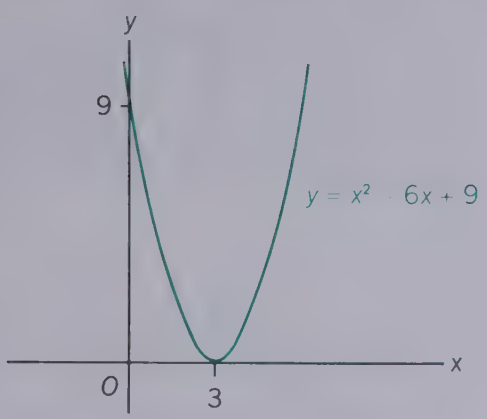
4. Mark y-intercept



5. Sketch graph

When the graph of a quadratic function just approaches the x-axis and touches it at one point before moving away we refer to it as having *two equal intercepts*. This is consistent with the solution for the zeros of the function.

$$y = x^2 - 6x + 9$$
$$0 = (x - 3)(x - 3)$$
$$x = 3 \text{ or } x = 3$$



EXERCISE 9·4

1. Draw the graph of the quadratic function $y = x^2 - 4x + 7, -1 \leq x \leq 5$.

   (a) What is the minimum value of the function?

   (b) At what value of x does the minimum occur?

   (c) Find the y-intercept of the graph.

   (d) Does the graph have x-intercepts?

   When a graph of a quadratic function has no x-intercepts we refer to it as having imaginary intercepts. This will have more meaning when we study the algebraic solution of a quadratic equation.

2. Draw the graph of the quadratic function $y = -x^2 - 6x - 9, -6 \leq x \leq 0$.

   (a) What is the sign of the coefficient of x^2? Of what significance is this sign with respect to the graph?

   (b) What is the equation of the axis of symmetry?

   (c) What is the maximum value of the function?

   (d) Give the coordinates of the vertex.

   (e) How many intercepts does the graph indicate? What are they?

   (f) Check the x-intercepts by solving for the zeros of the function. Do they agree with the graph?

3. For each of the quadratic functions

   (i) $y = x^2 - 5x$ (ii) $y = -x^2 + x + 6$
   (iii) $y = x^2 + 10x + 25$ (iv) $y = 2x^2 - x - 10$
   (v) $y = x^2 + 2x - 15$ (vi) $y = 3x^2 + 2x - 16$
   (vii) $y = x^2 - 4x + 16$ (viii) $y = 5x^2 + 24x - 5$

   determine:

   (a) the x-intercepts of the graphs (if real)

   (b) the axis of symmetry

   (c) the maximum or minimum value

   (d) the y-intercept.

   Then sketch a graph for each.

9·5 Problem Solving

Example 1

A paint store was selling 80 gallons of paint a month (on the average) when the selling price was \$10 per gallon. Every time the selling price was decreased by \$1.00 per gallon the sales increased by 10 gallons per month. At what price per gallon would the dollar sales be maximum for the month?

Solution

Let y represent the dollar sales for the month.

Let x represent the number of \$1 decreases per gallon.

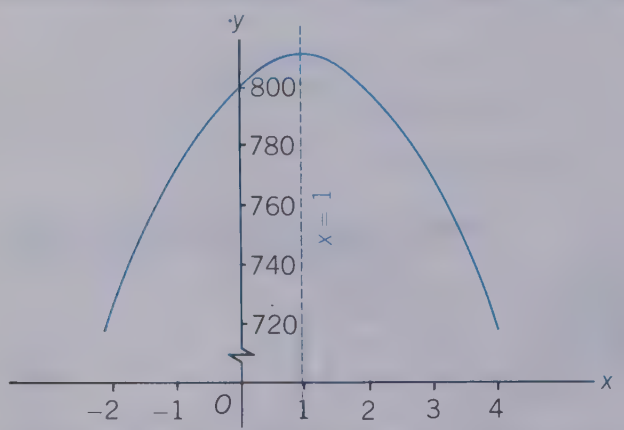
Then y = (price per gal) (total gal sold)

$$y = [10 - x(1)]\,[80 + x(10)]$$

$$y = 800 + 20x - 10x^2$$

Table of values for graphing:

x	-2	-1	0	1	2	3	4
y	720	770	800	810	800	770	720



The maximum is $y = 810$ when $x = 1$ (axis of symmetry).

∴ The maximum sales for the month are \$810 when the price per gallon is \$(10 − x), i.e., \$9.

How many gallons was the store selling at \$6.00 per gallon?

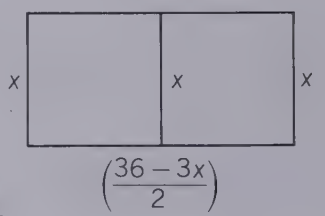
Do this by another method and check your answer with that given here.

Example 2

Jim has 36 feet of screen which he wants to use to build an enclosure for his pet rabbits. He intends to enclose a rectangular plot and divide it into equal areas. What is the maximum area he can enclose? What are the dimensions of the maximum area?

Solution

Let the width of the plot be x feet. There will be three such lengths of fence.



The remainder of the fence will be $(36 - 3x)$ feet.

But this is for the two longer sides. Therefore, each length of the whole plot will be $\left(\frac{36 - 3x}{2}\right)$ feet.

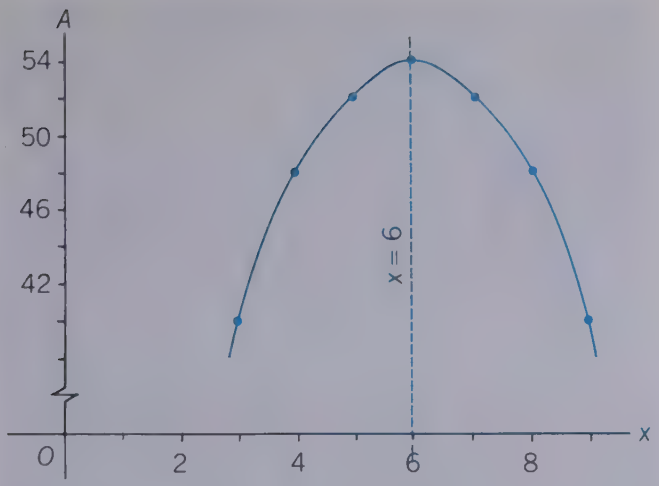
$$\text{Area} = lw = \left(\frac{36 - 3x}{2}\right)x$$

$$= \left(18 - \frac{3x}{2}\right)x$$

$$= 18x - \frac{3}{2}x^2$$

Table of values for graphing:

x	3	4	5	6	7	8	9
A	$40\frac{1}{2}$	48	$52\frac{1}{2}$	54	$52\frac{1}{2}$	48	$40\frac{1}{2}$



The maximum is $A = 54$ square feet,
when $x = 6$ feet.

Therefore the length is $\frac{36 - 3(6)}{2} = 9$ feet, when width = 6 feet.

Do this question another way (algebraically) and check your answer with that given here.

EXERCISE 9·5

Use the method you like best to solve these.

1. Susan has 18 feet of ornamental border to enclose a flower plot. She wishes to design a rectangular plot of maximum area to be enclosed by the 18 foot border. What are the dimensions of the flower plot? What is the maximum area? (Use the quadratic function concept to solve this).
   ANSWER $4\frac{1}{2}' \times 4\frac{1}{2}'$ $20\frac{1}{4}$ sq feet
2. Terry has 12 feet of screen. He is designing an enclosure for his pet rabbit. It is to be a rectangular plot against his father's garage. He wants the largest area possible in this enclosure using the garage wall as one side of it. What will the area of the enclosure be? What will be its dimensions?

ANSWER: 18 sq ft. 3' x 6'

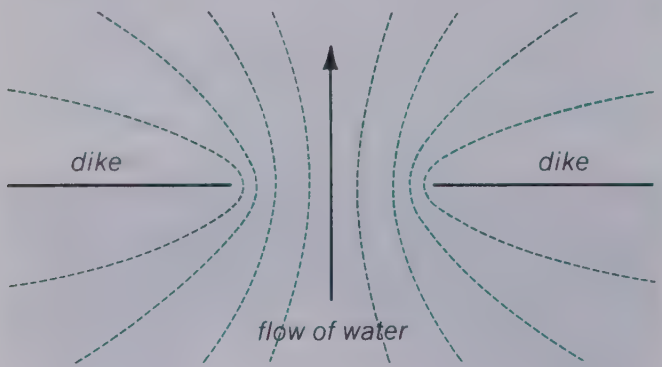
3. Billy was an enterprising young business man. He started out selling lemonade on the sidewalk at 9 cents a glass. At that price, he averaged 10 glasses per day. He noted that when he decreased the price by 1 cent a glass his average sales increased by two glasses per day. What price per glass should he charge to achieve maximum sales?
   ANSWER: 7¢
4. Find two positive real numbers whose sum is 11 and whose product is a maximum.
   ANSWER $5\frac{1}{2}, 5\frac{1}{2}$
5. A graphic arts worker is given the following order: Design a card with printed part 32 inches in perimeter, margins on top and bottom of 3 inches each, margins on the sides of 2 inches each, area of card to be a maximum.
   What will the maximum area be? 13" X 13"
   What are the dimensions of the card? 169

6. The second element of an ordered pair is 3 times the square of the first minus 12 times the first. What is the minimum value of the second element for this relation? What is the corresponding value of the first element?

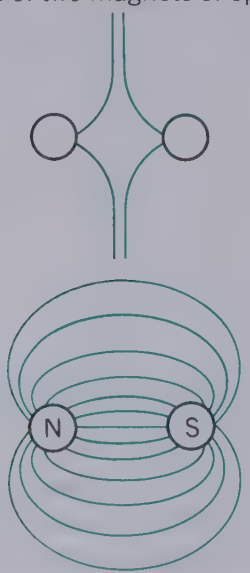
$-12, 2$

7. Solve the three questions in Section 9·1.

8. Quadratic equations can be used to express algebraically the two-dimensional flow of water. Can you identify parabolic and hyperbolic current flows?



9. The fields of force around two magnets are shown. The first is of two magnets of the same polarity. The second is of two magnets of opposite polarity.



Use iron filings on a sheet of graph paper with the magnets beneath it to see if you get these patterns. Do you think these lines are related to the conic sections?

Using the Library

1. The seventeenth century reflects the reawakening of the world after a period of relative dormancy. Continue the date line by placing these events and names on the chart.

Fermat, born – lawyer who studied math as a hobby
John Milton, born
Champlain sailed up the St. Lawrence
Telescope invented
Kepler published laws of planetary motion
Shakespeare died
Wallis could calculate square root of a 53 digit number in his head
Napier published work on logarithm
Pilgrims landed at Plymouth
Pascal born – probability
Huygens, born – improved lenses
Oughtred – slide rule
Sir Isaac Newton, born
Leibniz, born
Halley, born
Charles II becomes King of England
De Moivre, born
St. Paul's Cathedral designed and started
Bach and Handel, born
Captain Kidd buried treasure
Halley's comet passes near Earth

2. Fermat wrote "Every prime of the form $4n + 1$ can be expressed as the sum of two squares." Sample a number of such primes to see if you support this statement.

3. Comment on Newton, the man, his personality, his abnormal fear of controversy, his childish behavior reflected by his comment "going to publish no more" after some harsh criticism, and his refusal to enter into debate about his writings.

   What did Newton mean when he said "I do not know what I may appear to the world; but to myself I seem to have been only like a boy, playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."?

   Discuss how important it is what others think of you as compared to what you think of yourself.

   Which do you think Newton felt was the most important?

4. Prepare a scientific report on Kepler's three laws of planetary motion.

5. Leibniz, Newton and Huygens (sometimes spelled Huyghens), amongst others, studied the brachistochrome. What is the brachistochrome? How is it related to a cycloid?

REVIEW EXERCISE 9

1. For the function defined by $y = 6x^2 + 19x + 10$ complete the following table of values.

x	-4	-3	-2	-1	0	1
y						

   (a) On appropriately labelled axes plot the points from the above table.

   (b) From the plot in (a) can we accurately determine the x-intercepts and the maximum or minimum value?

   (c) Use the algebraic method (including formulae) to determine the x-intercepts and the maximum or minimum value.

   (d) What is the axis of symmetry?

2. On another planet an object projected upwards achieves a height (h) in time (t). If the object obeys the law defined by the function

$$h = 168t - 24t^2$$

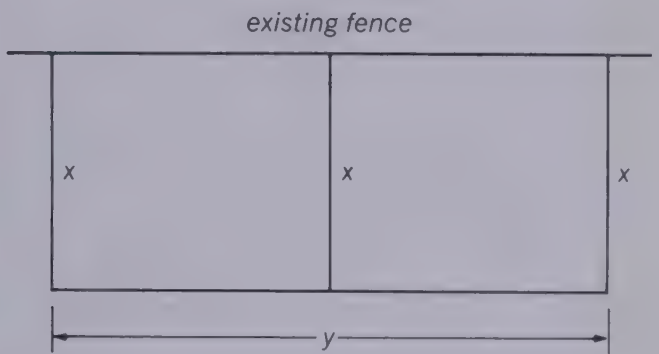
   determine:

   (a) the maximum height reached,

   (b) the time (t) when the maximum h occurs,

   (c) the total time of flight.

3. A farmer uses 96 yards of fencing to enclose an area against an existing fence. The enclosed area is to be divided into two equal parts. What will the dimensions of one part be if the total area enclosed is maximum?



Your Job and Mathematics

This project arose in the author's mind following a discussion with the owner of a "mixed farm". What do we mean by a "mixed farm"?

Project

Project 1

Purpose

To prepare a prospectus for a secondary industry to supplement a small farm income.

Notes: Very often the small farmer supplements the farm income by such small industries as raising chickens both for egg sales and for selling chickens for meat. Also many farmers keep bees for honey production. If facilities are available or can be obtained many farmers will supplement their income by hog production.

It is not unusual for the chicken raising to be the responsibility of the housewife – who also cares for the house, prepares the meals, cares for the children, and handles a multitude of other jobs.

Before entering into any of the secondary industries, the farmer and his wife should analyse the business endeavor. For example, certain expenses must be costed out, such as the cost of buildings if new ones are needed, cost of feed, supplements, sale of product services to the buildings, fences, etc. The expected income must be estimated. Also the interest on the investment must be calculated. Do you think a farmer and his wife should calculate as expenses his/her wages for time spent?

Method

Write your department of agriculture for information and a guide to farming covering:

(a) poultry production

(b) hog production

(c) any others that you wish.

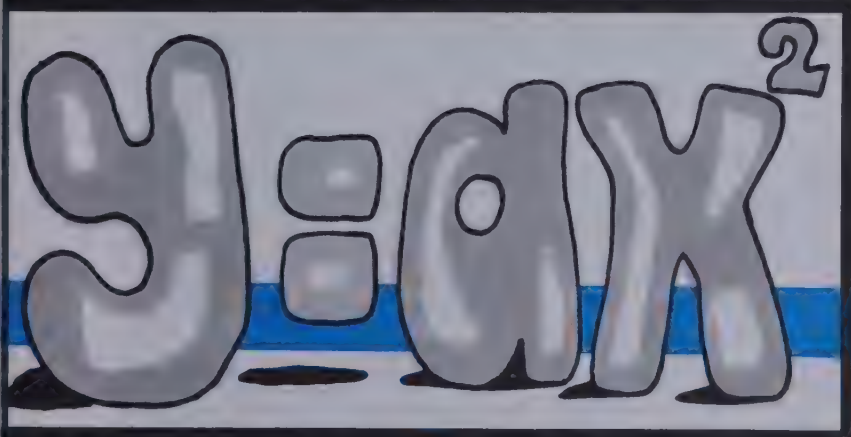
The publication *Alberta Farm Guide* is available in single classroom copies (apply on school letterhead) from: The Alberta Department of Agriculture, Edmonton, Alberta. The Canadian Department of Agriculture, Ottawa has similar publications.

Girls: Pretend you are a wife on a small farm which grows wheat. Using information from the booklets or guides to farmers and current market prices, analyse the cost of and the income from the raising of chickens for both meat and eggs. Do you think this would be a profitable endeavor? How many chickens would be necessary to break even?

Boys: Pretend you are a farmer on a small farm which grows wheat, oats and barley. Obtaining information from the booklets mentioned and from the current local market prices, analyse the cost of and the income from the raising of hogs as a secondary industry. How many hogs would be necessary to break even? Do you think this would be a profitable endeavor?

Chapter Ten

THE PARABOLA



10·1 Introduction

The parabola, apart from being an interesting mathematical curve, is also a very useful one in many applications. One of the applications is in design and drawing. We shall attempt to learn about the curve by two methods of drawing it.

Projects

Project 1

Purpose

To learn to design a parabola by an interesting and colorful approach.

Method

On a firm cardboard draw two line segments to form an angle. (See Figure 1.) Mark off the segments in equal divisions, keeping in mind that the number of divisions on each segment is the same.

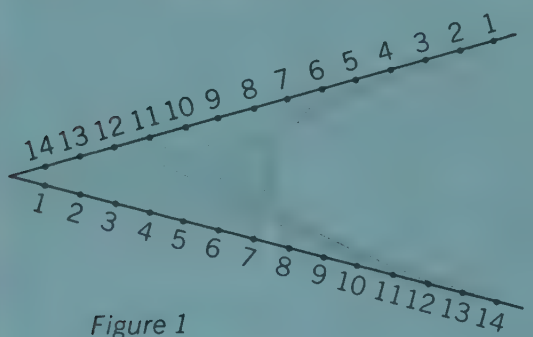


Figure 1

Punch holes to mark off the divisions. In Figure 1 the holes are numbered for instructive purposes only. Once the holes are prepared use colored string or yarn for the stitching. The string should be so laced as to join hole 1 to hole 1, hole 2 to hole 2, hole 3 to hole 3, etc., until completion.

Questions

1. What would be the effect on the curve if the holes were very wide apart? very close together?

   Hint! Discuss the smoothness of the curve produced.

2. Can the rays on which the holes are punched be at a different angle to each other and still produce a curve affect?

3. Can the rays be parallel to each other and still produce a curve effect? Why?

4. The above method is called *curve-stitching*. If, instead of using string or yarn, straight lines were drawn with the aid of a ruler, would a curve effect be produced also?

The curve produced by the curve-stitching is a parabola.

Project 2

Purpose

To construct a parabola by a mechanical method.

Method

Obtain a T-square, thumbtacks, string, a pencil and some tape.

Tape down or tack a large sheet of paper on a table top so that one edge of the paper is coincident with the edge of the table. Draw the axes for the parabola as in Figure 1. Tack a string to the end of the T-square at *C*. Stretch the string firmly along the T-square and measure as if to tack it at a point marked *B*. With segment *BC* as measure of the string to use, transfer the end *B* to point *F* on the axis and tack down the string at *F*. Place the pencil at *P* along edge of T-square so that string is always taut.

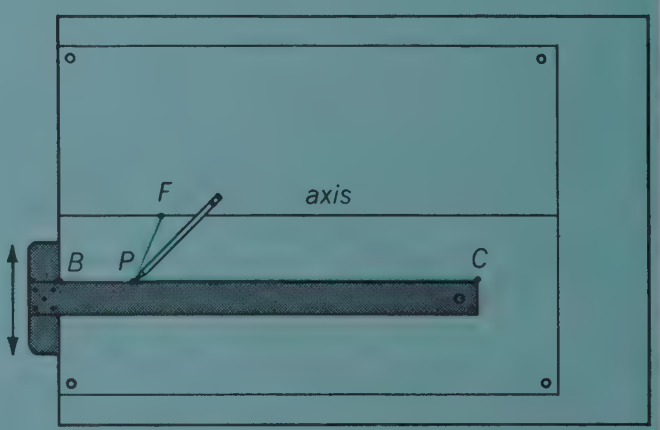


Figure 1

The operation requires coordination in the movements of the pencil and T-square. While the T-square is slowly sliding along the edge of the table, the pencil is tracing a curve. The pencil must always keep the string taut. (When finished with lower half of the curve make an adjustment to continue the upper half.)

Questions

1. From the drawing, for any point P on the curve what is the relation between PF and PC?
2. For any point P on the curve what is the relation between PF and PB?

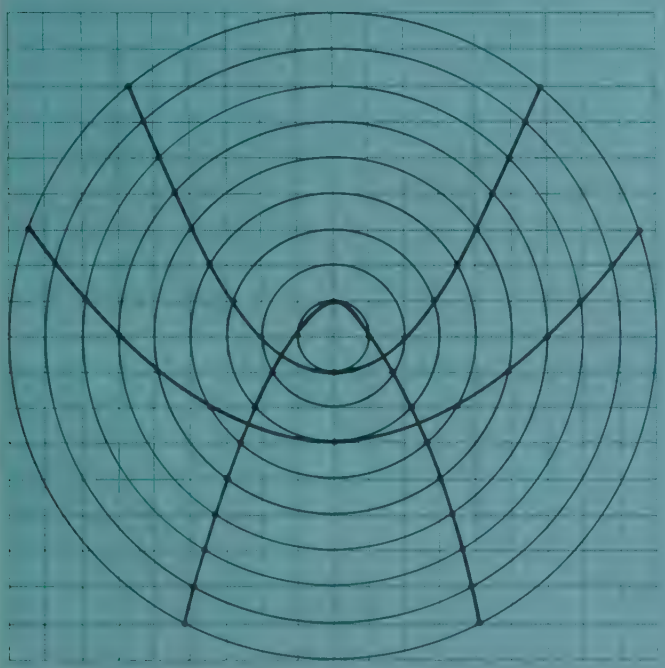
Project 3

Purpose

To observe the parabolic curve in various designs.

Method

Study Figure 1. Do you know how the diagram was constructed? Can you determine the pattern for locating the points which outline the curve?



How many patterns similar to the outlining curve can you find in this drawing?

Figure 1

Draw another basic diagram as in Figure 1. Observe the shaded portions of Figure 2. Do you see the appearance of the curve emerging due to the shading? On your diagram try other shadings to produce similar curves.

Figure 2

Project 4

Purpose

To draw a parabola using the rule: "A parabola is the locus of a point such that its perpendicular distance from a given line is congruent to that distance from a given point not on the line."

Theory

We shall use reflections to do this project.
Obtain a piece of colored plastic reflector about $\frac{1}{8}''$ thick and $2'' \times 3''$. Red is the best color. You must be able to see *through* the plastic. Construct a wooden block as shown to hold the plastic *vertical* to the *desk*. We will call this instrument *Reflecto*.

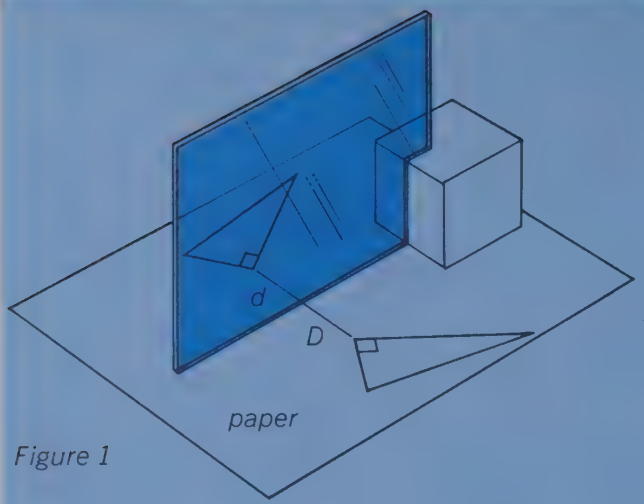


Figure 1

Remember! The distance the image (reflection of the object) is behind the plastic reflection (mirror) is the same as the distance the object is in front of the plastic reflector.

measure of d = measure of D

(We will use the *Reflecto* again in Chapter 11.)

Method

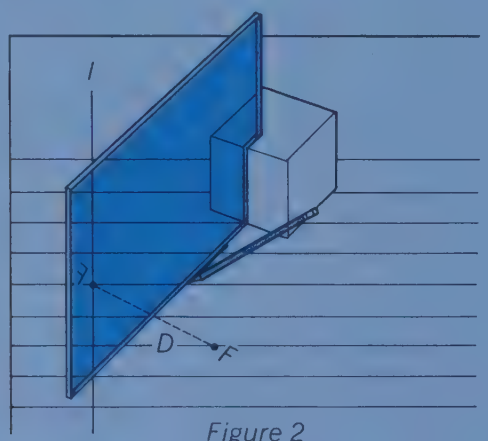


Figure 2

1. On a sheet of narrow ruled paper draw a line *l* perpendicular to the rules and close to one edge.
2. About 1 inch from line *l* place a point *F* on one of the ruled lines (as shown).
3. Place Reflecto on the page between the line *l* and point *F* so that the image of point *F* lies on the intersection point of the line *l* and a ruled line.
4. Draw a short segment ($\frac{1}{4}$ inch long) at the Reflecto's edge on *the ruled line*.

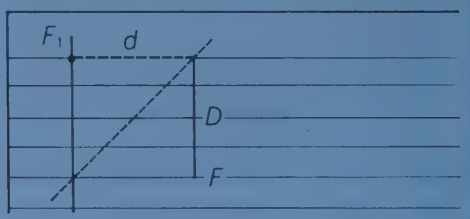


Figure 3

5. Notice in Figure 2 that $m(d) = m(D)$. This is according to the rule.
6. Continue to move the plastic and repeat 4 for other ruled lines.

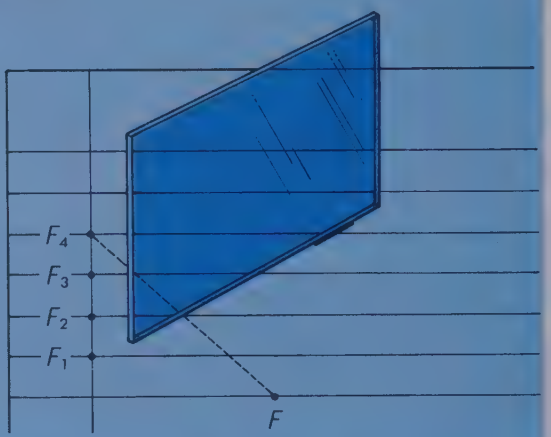
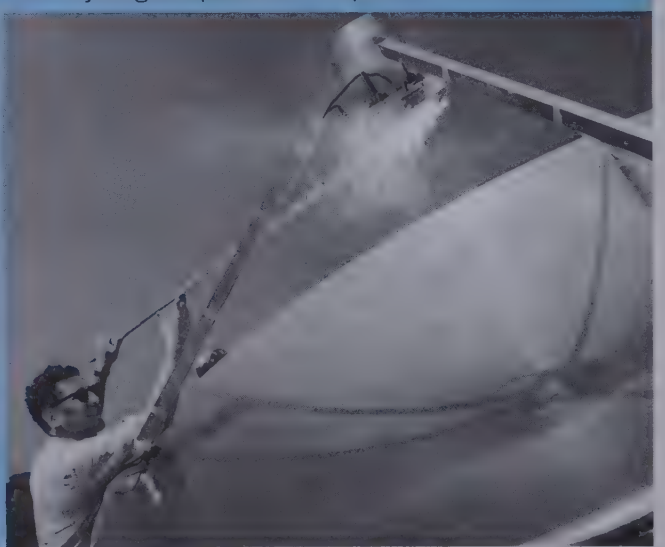


Figure 4

7. Do you get a parabola? Explain.



Parabolic curves are used in this solar-powered thermo-electric system which converts the heat of sunlight directly into electricity.

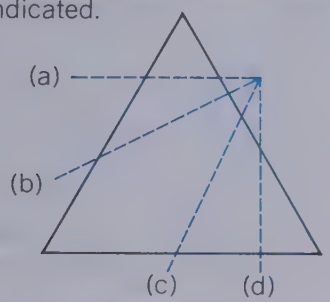
In Chapter 9 on Quadratic Functions, the graph of the function was called a parabola. There are many occurrences in our physical world where the parabolic curve may be observed. Some of these occurrences are: reflectors in headlights, some arches in bridges and aqueducts, some arches in buildings, some turns in highways.

10·2 Conic Sections (Curves of the Cone)

A very interesting study can be made of the various curves that one finds in cutting a slice through a cone.

Demonstration Activities

Obtain a large styrofoam cone (if necessary, fashion one from a cube of styrofoam). Then use a sharp knife to cut the cone as indicated.



Side view of cone resting on circular base

Figure 1

Cut (a) parallel to base
(b) at an angle between (a) and (c)
(c) parallel to inclined slope of cone
(d) perpendicular to base.

The names of the curves produced by these cuts are:

(a) circle (b) ellipse
(c) parabola (d) hyperbola.

Place the cut surfaces on paper and trace around cut edge to obtain the curve.

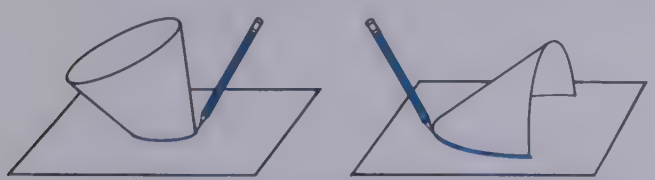


Figure 2

Here is another activity worth your consideration. In the middle of a large cardboard cut out a 4-inch circle. Mount the cardboard about one foot away from a plain wall or tackboard. Then with a clear (unfrosted) light bulb in positions shown in Figure 3 project the cut out circle on the wall.

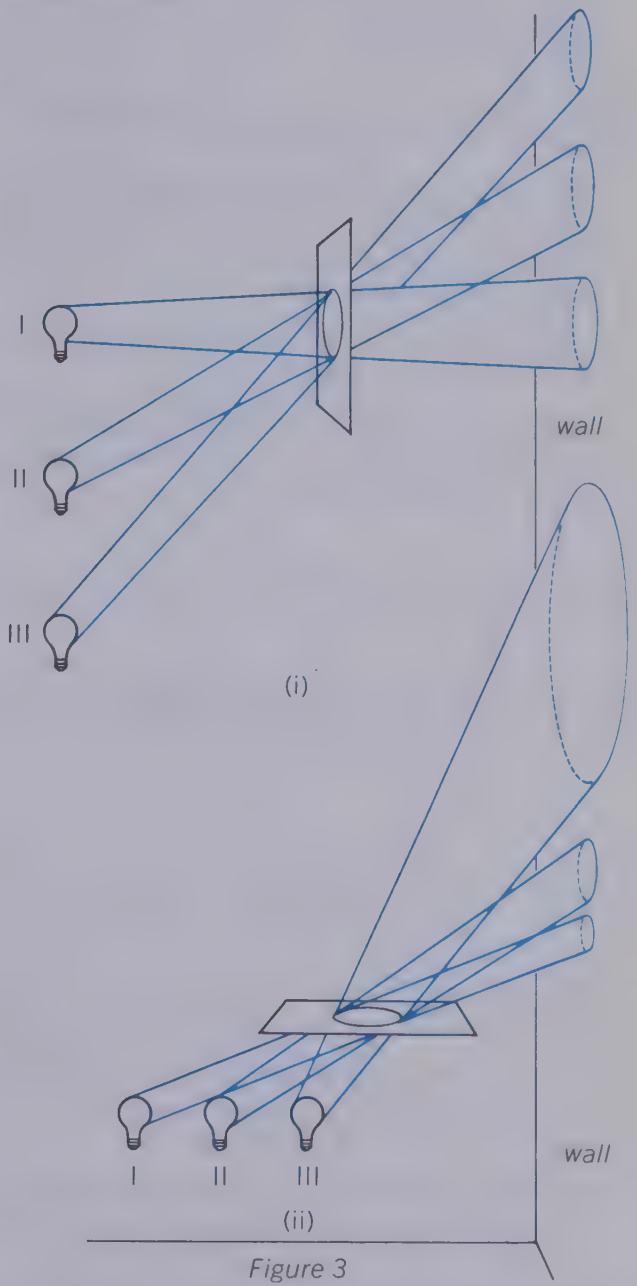


Figure 3

Tack a sheet of paper on the wall and trace the curve(s) projected on it for further study.

10·3 Parabolas

A parabola is a set of points in a plane such that each point is equidistant from a stationary position (called the focus) and a straight line (called the directrix).

With the above definition as a guide, a geometric construction of a parabola can be executed.

1. On a sheet of ruled paper, draw a straight line perpendicular to the ruled parallel lines and dividing the sheet in two equal parts. This line will be the axis of the parabola.
2. On the left side of the sheet choose one of the ruled lines as the directrix and mark it.
3. Four divisions to the right of the directrix and on the axis locate the focus F. Your sheet should now appear as in Figure 10-1.

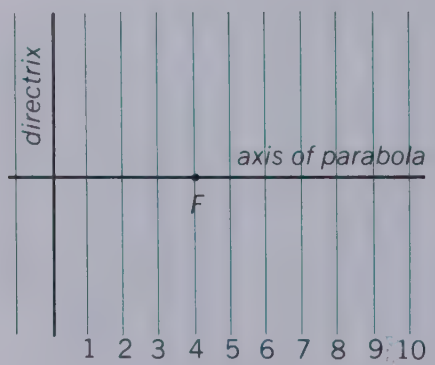


Figure 10-1

4. From the directrix number the ruled lines to the right. Line 4 should have F on it.
5. Halfway between the directrix and F mark the vertex V on the axis. The vertex is a point of the parabola where it meets the axis.
6. With compasses point on F and radius equal to three divisions of ruled paper, cut arcs on line numbered 3 at A_1 and A_2 as in Figure 10-2.
7. With compasses point on F and radius equal to four divisions of ruled paper, cut arcs on line numbered 4 at B_1 and B_2 as in Figure 10-2.

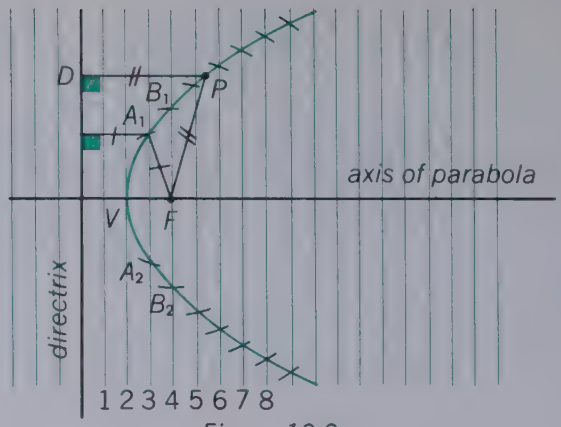


Figure 10-2

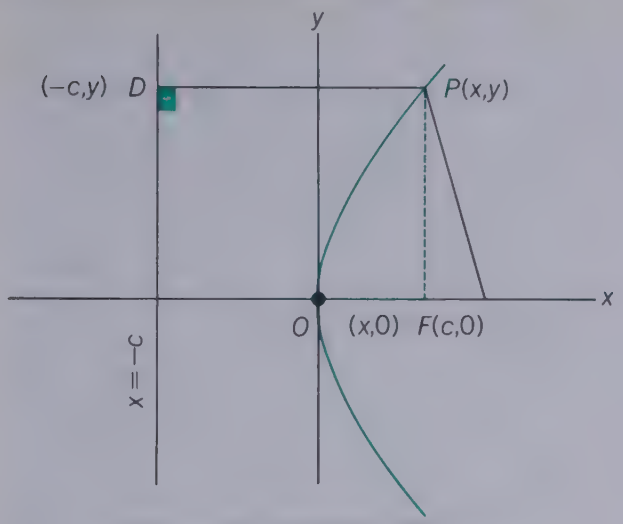
8. Continue 6 and 7 only adjusting the radius to the appropriately numbered line.
9. Join all the points cut off by the arcs into a smooth curve. The curve is a *parabola*.
10. Choose any point P on the parabola. Join PF and draw PD, perpendicular to directrix (D is a point on the directrix). What is the relation between PF and PD?
11. On other sheets of ruled paper repeat the above constructions for parabolas where the point P is 8 divisions from the directrix; 2 divisions from the directrix.
    Observe the three constructions for similarities and for differences. Discuss these.

$$y^2 = 4cx$$

10·4 Equation of a Parabola: $y^2 = 4cx$

We will now develop an equation for the parabola which has its vertex at the origin of a Cartesian plane and the x-axis as the axis of the parabola.

Let us study the sketch of such a parabola.



$m(\overline{PF}) = \sqrt{(x-c)^2 + (y-0)^2}$ *Why?*

$m(\overline{PD}) = c + x$ *Why?*

Figure 10-3

Choose any point $P(x,y)$ of the parabola. Let the coordinates of the focus be $(c,0)$, that is, the distance VF is c units. Similarly, the distance from the directrix to the vertex at the origin is c units. (Why?) Therefore, the equation of the directrix is $x = -c$.

Join PF and from P draw a perpendicular to the directrix at D.

The coordinates of D will be $(-c,y)$. Why?

Since $m(PF) = m(PD)$, (By what authority?)

$$\sqrt{(x-c)^2 + (y-0)^2} = \sqrt{[x-(-c)]^2 + (y-y)^2}$$

(Applying distance formula)

$$(x-c)^2 + y^2 = (x+c)^2 \quad \text{(Squaring both sides)}$$

$$x^2 - 2xc + c^2 + y^2 = x^2 + 2xc + c^2$$

$$\boxed{y^2 = 4cx}$$

This is an equation of the parabola, with vertex at the origin and F a point of the x-axis.

> The defining equation for the parabola
> (i) with vertex at the origin,
> (ii) having the focus on the x-axis,
> (iii) with coordinates of $F(c,0)$,
> (iv) having a directrix $x = -c$, $c > 0$
> is $y^2 = 4cx$.

Example 1

For each of the parabolas

(i) $y^2 = 8x$ (ii) $y^2 = -20x$ (iii) $x^2 = -12y$

identify the focus and the directrix.

Solution

Comparing $y^2 = 8x$ with $y^2 = 4cx$ we note $4c = 8$ and hence $c = 2$.

(i) The focus of a parabola is given by $F(c,0)$.
Therefore, the focus is $F(2,0)$.
The directrix of a parabola is given by $x = -c$.
Therefore the directrix in this case is $x = -2$ or $x + 2 = 0$.

(ii) For $y^2 = -20x$, $4c = -20$ and $c = -5$.
Therefore the focus is $F(-5,0)$ and the directrix is $x = 5$ or $x - 5 = 0$.

(iii) For $x^2 = -12y$, $4c = -12$ and $c = -3$.
But this parabola has its focus on the y-axis. (Why?)
Therefore, the focus is $F(0,-3)$ and the directrix is $y = 3$ or $y - 3 = 0$.

From tables of values sketch graphs of each parabola to verify each answer.

In Example 2 we will find the equation of the parabola given a point on a general parabola.

Example 2

A parabola of the form $y^2 = 4cx$ has $P(2,8)$ as a point of the graph. Give the equation of the parabola.

Solution

If $P(2,8)$ is a point of the graph, then when $x = 2$ the corresponding y-value is 8.
Substitute these values in $y^2 = 4cx$.

$$8^2 = 4c(2)$$
$$64 = 8c$$
$$\therefore \quad c = 8$$

The equation of the required parabola is

$$y^2 = 4(8)x$$
$$y^2 = 32x.$$

Sketch this parabola.

Objects thrown through the air may follow parabolic curves.

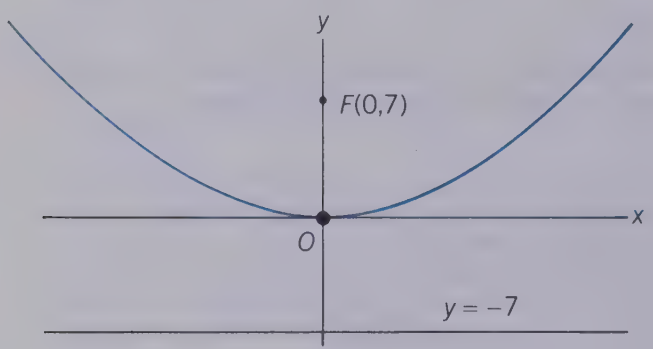
In Example 3 we are given the vertex and the directrix. We must find the equation of the parabola.

Example 3

A parabola has a vertex $V(0,0)$ and a directrix $y = -7$. Find the equation of the parabola.

Solution

A sketch is helpful.
Because the directrix is $y = -7$, the focus is on the y-axis.
Since the directrix is as far from the vertex as the focus is (by definition) then the focus is at $(0,7)$. Hence, $c = 7$.



The equation of the parabola must be of the form
$x^2 = 4cy$. (Why?)

Hence, the required equation is
$x^2 = 4(7)y$
$x^2 = 28y$.

Project

Project 1

Purpose

To investigate the relationships of four parabolas with focal points on each of the positive and negative x-and y-axes.

Method

Work in groups. Sketch the graphs of:

(i) $y^2 = 4x$
(ii) $y^2 = -4x$
(iii) $x^2 = 4y$
(iv) $x^2 = -4y$

on the same coordinate axes.

Discuss:

(a) What do (i) and (ii) have in common?
What do (iii) and (iv) have in common?
(b) In what way(s) do (i) and (ii) differ?
Is there a common axis of symmetry for (i) and (ii)?
What is the axis of symmetry?
(c) Is there a common axis of symmetry for (i) and (iii)? What is it?
(d) How many axes of symmetry are there for the four graphs on one set of coordinate axes?
Name them.
Hint! There are more than two.
(e) Summarize in your notebook the things you learned.

EXERCISE 10·4

1. For each of the following what are the focus and directrix of the graph:
   (i) $y^2 = 24x$ (ii) $y^2 = -16x$
   (iii) $x^2 = 36y$ (iv) $x^2 = -72y$

2. If a parabola with a vertex at the origin has its focus at $F(-5,0)$ what is the equation of the parabola?

3. The graph of a parabola has the points $P(3,9)$ and $V(0,0)$. What is the equation of the parabola?
Can there be two parabolas?

4. The directrix of a parabola is $y + 3 = 0$. If the vertex is at the origin, what is the equation of the parabola?

5. Find the defining equation of each of the following parabolas:
   (a) $V(0,0)$ and $F(0,4)$
   (b) $V(0,0)$ and directrix $2x + 3 = 0$
   (c) $V(0,0)$, focus on y-axis and passing through $P(-4,-6)$
   (d) $V(0,0)$, focus on x-axis and passing through $(15,25)$.

10·5 Solution of a Linear-Quadratic System of Equations

A straight line is the graphical representation of

$$y - 3x = 6.$$

A parabola is the graphical representation of

$$y^2 + 24x = 0.$$

The points where the two graphs intersect are the solutions. This means that the x-and y-values at each point of intersection will make each equation a true statement.

Example 1 illustrates a graphical solution to a system of equations involving a linear equation and a quadratic equation. Then an algebraic solution is given.

Example 1

Solve the following system of equations
(i) graphically, (ii) algebraically.

$$y^2 + 24x = 0$$
$$y - 3x = 6$$

Solution

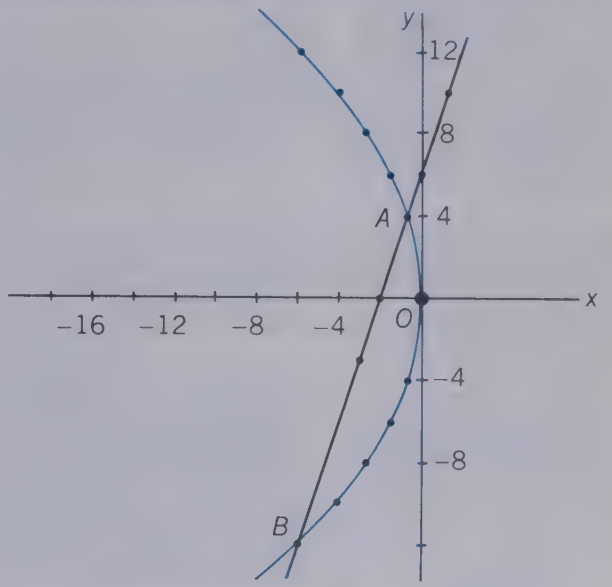
First graph the two equations:

$y^2 + 24x = 0$ $\qquad$ $y - 3x = 6$

$y^2 = -24x$

x	0	$-\frac{2}{3}$	$-\frac{3}{2}$	$-\frac{8}{3}$	-6
y	0	± 4	± 6	± 8	± 12

x	0	-2	-3	1
y	6	0	-3	9



From the graph we read the solution set as the points of intersection of the two lines.
They are $A(-\frac{2}{3},4)$ and $B(-6,-12)$.

Check the solution set A by replacing the respective values in the two given equations.

Check the solution set B by replacement.

An algebraic solution to the same problem is as follows:

$$y^2 + 24x = 0 \quad (1)$$
$$y - 3x = 6 \quad (2)$$

Express x in terms of y (or y in terms of x) and then substitute in (1).

From (2) $\qquad 3x = y - 6$

$$x = \frac{y - 6}{3} \quad (3)$$

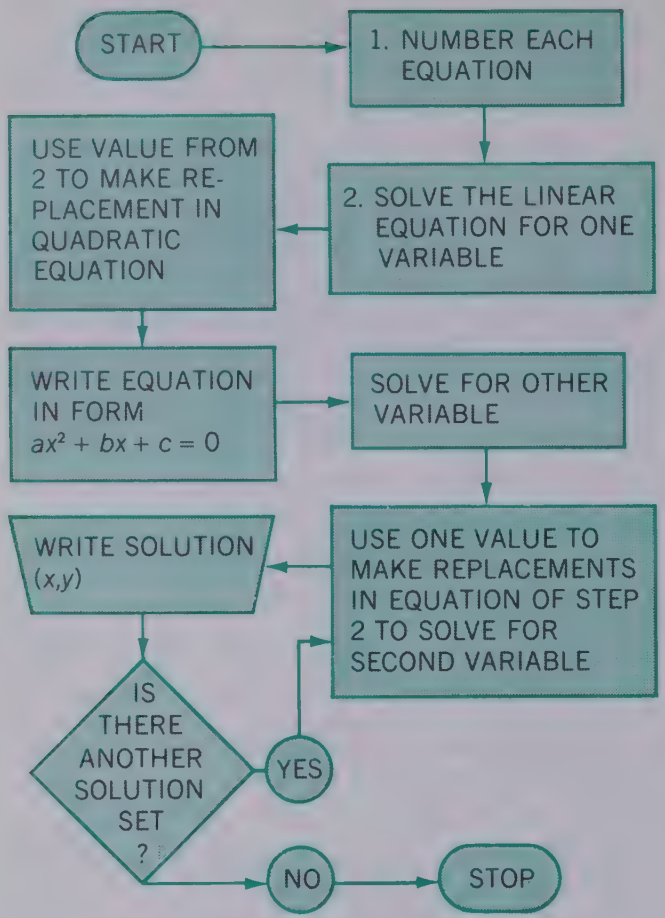
Substitute (3) into (1)

$$y^2 + 24\left(\frac{y - 6}{3}\right) = 0$$
$$y^2 + 8y - 48 = 0$$
$$(y + 12)(y - 4) = 0$$
$$y + 12 = 0 \text{ or } y - 4 = 0$$
$$y = -12 \text{ or } y = 4$$

into (3) $\qquad x = \frac{-12 - 6}{3} \qquad x = \frac{4 - 6}{3}$

$$= -6 \qquad = -\frac{2}{3}$$

Solution set is $\{(-6,-12), (-\frac{2}{3},4)\}$.


START
1. NUMBER EACH EQUATION
2. SOLVE THE LINEAR EQUATION FOR ONE VARIABLE
USE VALUE FROM 2 TO MAKE REPLACEMENT IN QUADRATIC EQUATION
WRITE EQUATION IN FORM $ax^2 + bx + c = 0$
SOLVE FOR OTHER VARIABLE
USE ONE VALUE TO MAKE REPLACEMENTS IN EQUATION OF STEP 2 TO SOLVE FOR SECOND VARIABLE
WRITE SOLUTION (x,y)
IS THERE ANOTHER SOLUTION SET ?
YES
NO
STOP

For Discussion

For a system of equations as in Example 1 there are two solutions if the graphs intersect in two points.

1. When do you think there is only one solution?
2. When do you think there is no solution?

EXERCISE 10·5

1. Solve the following systems of equations graphically. Check (a), (b), (d) by algebraic solution.

   (a) $x^2 = y$
   $2x - y = -3$

   (b) $x + y^2 = 0$
   $y - 3 = 0$

   (c) $x^2 = -4y$
   $x + y = 4$

   (d) $y = x^2 - 3x$
   $2x + y = 6$

2. Review again the two questions in the preceding For Discussion.

3. Solve graphically the following systems of equations then check by algebraic solution where possible.

   (a) $y^2 = 12x$
   $2x - y = 12$

   (b) $x^2 = -9y$
   $x + 3y = 0$

   (c) $x^2 = 16y$
   $x - 2y = 2$

   (d) $y = 3x^2 + 7x - 6$
   $x + 3 = 0$

   (e) $y = x^2 + x - 12$
   $5x - y = 12$

   (f) $y = 2x^2 - 7x + 5$
   $3x + y = 11$

   (g) $y = x^2 - 4x - 12$
   $2x - y = 21$

   (h) $x - 4y^2 = 0$
   $3x + 4y - 8 = 0$

   (i) $y^2 = 6x$
   $x = -2$

   (j) $x^2 = 16y$
   $y = -1$

10·6 Applications of Parabolas

The parabola has many applications in the manufacture of parabolic reflectors like those used in automobile headlights, searchlights, huge telescopes, radio antennas, microwave antennas, etc. Other uses of parabolas, or near parabolas, are arches of bridges, doorways and ceilings of buildings such as cathedrals. The path of a projectile is a parabola.

The unique property of a parabolic reflector is illustrated.

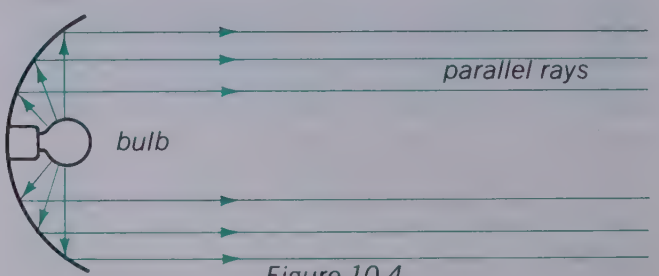


Figure 10-4

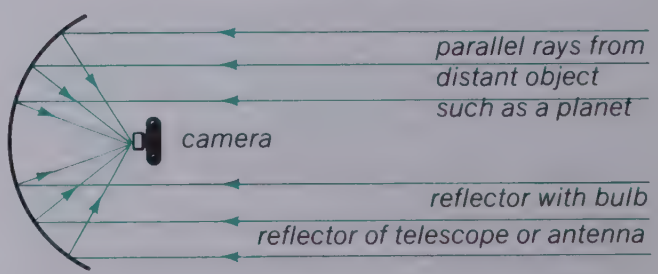


Figure 10-5

In Figure 10-4, light emitted from a light bulb or an arc (searchlight) located at the focus of the parabola, is radiated in all directions. The rays that strike the reflector are redirected in parallel rays away from the reflector. This is why a searchlight can pierce deep into darkness with a concentration of rays (beam) of light.

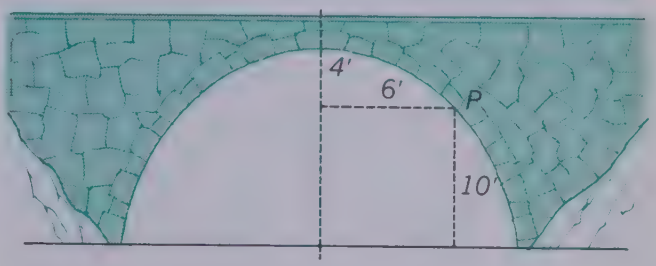
In Figure 10-5, the reflector operates in reverse. All parallel rays of light which enter the reflector parallel to the axis of the parabola are then concentrated at the one point, the focus. This property is of invaluable use in telescopes and (dish) antennas.

Arches that are near parabolic are used in constructions extensively because of their special strength characteristics. Examples of these kinds of arches may be found in bridges and buildings (such as churches).

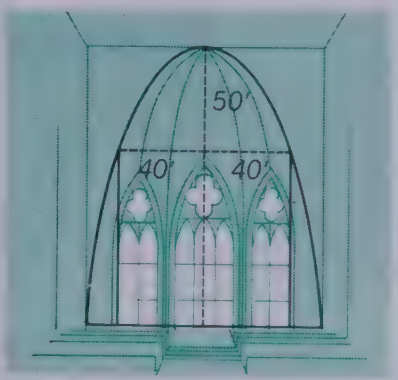
EXERCISE 10·6

For the following the vertex of each parabola is at the origin of the Cartesian plane.

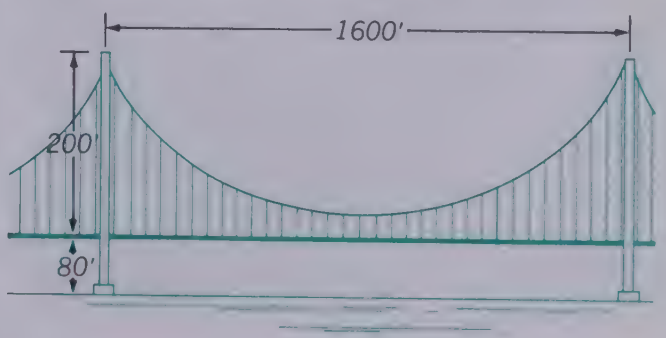
1. The bridge arch shown is assumed to be a parabolic curve. One point P is marked and the dimensions shown are recorded. Calculate the equation of the arch.



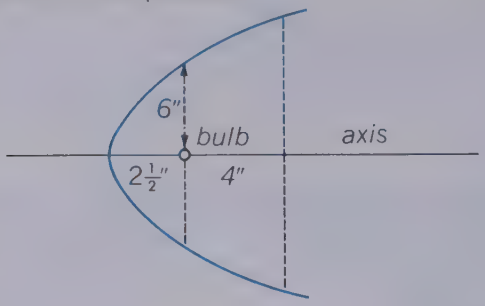
2. Assuming the arch in the ceiling of this cathedral to be a parabolic curve, calculate the equation of the arch from the given information.



3. Assume the cable in the bridge forms a parabolic curve and the arch is parabolic. Calculate the equation for each parabolic curve.



4. A designer wishes to construct a reflector such that a bulb placed $2\frac{1}{2}$" from the vertex of the reflector (on its axis) will produce parallel rays from the reflector. Use the data given to determine the equation of the parabola.



5. A contractor found this plan for a church window in the blueprints. Some measurements were missing although the equation of the parabolic curve was given. Can you find the widths of the windows shown by *A*, *B*, and *C*? Round all answers to one decimal place.

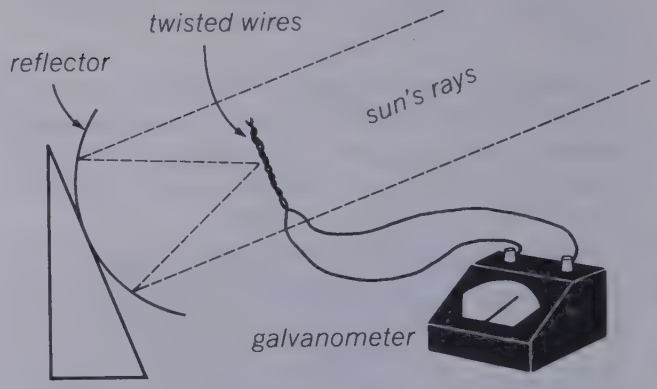
6. It is known that the equation of the parabolic path traced by a cannon shell (in miles) is

$$x^2 = -0.4y.$$

The spent cannon shell fell one mile from the cannon. The airplane is 2000 feet high. Did the cannon shell go high enough to hit the plane?

7. A radio telescope is 82 feet in diameter. If the depth of the dish is 22 feet and is parabolic in shape, what is the equation of the curve of the cross-sectional cut through its centre?

8. Parabolic reflectors are used to direct the sun's heat onto the thermocouple that generates electricity when heated.
   (a) Draw two diagrams to illustrate how each reflector can be used for this.
   (b) Obtain a parabolic reflector, a copper wire, a soft iron or aluminum wire, and a sensitive galvanometer. Set up as shown. Adjust the reflector to focus the sun's rays on the twisted wires. What do you observe?



   (c) What do you see as the future of this type of electricity generator in
      (i) Canada?
      (ii) Sahara region of Africa?
      (iii) Los Angeles, U.S.A.?
      Explain your answer to each.

Using the Library

1. The 18th century fairly erupted with mathematical discoveries. Place these names and events on your own time line and illustrate each event.

   Euler born
   Peter the Great attacks Persia
   Haydn born
   Saccheri born – forerunner of non-Euclidean geometry
   La Grange born – helped establish metric system in France
   Montcalm defeated by Wolfe on the Plains of Abraham
   Laplace born
   Mozart born
   Beginning of Industrial Revolution
   Comte de Buffon – calculated π by probability
   Watt invented steam engine
   Beethoven born
   United States – Declaration of Independence
   Gauss born
   United Empire Loyalists arrive in Canada
   Poncelet born
   French Revolution
   Cauchy born – one of Napoleon's engineers
   Lobachevsky
   Marie Antoinette guillotined
   Napoleon rules France
   Babbage, C.
   Sir Alexander Mackenzie reaches the Pacific Ocean

2. When Euler was introduced to King Frederick the Great's wife, the queen, he conversed in monosyllables. When asked why one day, he replied "Madam, it is because I have just come from a country where every person who speaks is hanged." Where had Euler been and what was the political environment in that country at that time?

   It is interesting to note that Euler was blind for the last 17 years of his life. Yet he continued to dictate mathematics the whole time.

3. At the age of 3 Gauss corrected a mistake in his father's computation of a weekly payroll at the office. At the age of 10 he found the sum of the integers from 1 to 100 inclusive in a few seconds. How long would it take you? How did Gauss find the sum? (See *Men of Mathematics*.)

4. The name Gauss is applied to several areas which reflect this man's wide interest and achievements. The curve of error is called a Gaussian Curve; the unit for measuring the force of a magnetic field is the gauss. What is meant by "degaussing" the metal hull of a ship?

5. Euler is noted for his solution to the Konigsberg (sometimes spelled Knenigsberg) Bridge problem. Research this problem. Can you "solve" it?

6. The dome of St. Paul's Cathedral in London is noted for the way it acts as a sound reflector. St. Peter's in Rome is well known for its whispering gallery. If someone speaks quite quietly at a certain point, the words can be heard quite clearly some distance away at a definite spot.

   Research these two cathedrals to find out how they are built in order to portray these characteristics.

REVIEW EXERCISE 10

1. For each of the parabolas identify the focus point and the directrix.

   (a) $y^2 = 20x$ (b) $y^2 = -32x$ (c) $y^2 = \frac{1}{2}x$

2. What are the identifying properties of the parabola $x^2 = 24y$?
   (Refer to focus, axis of symmetry, vertex and directrix.)

3. For a parabola with the focus of the x-axis and vertex the origin, give its equation if it passes through the point $(-9,6)$.

4. The directrix of a parabola is $y - 5 = 0$. If the vertex is at the origin, what is the equation of the parabola?

5. Find the defining equation of each of these parabolas:

   (a) $V(0,0)$ and directrix $x - 7 = 0$

   (b) $V(0,0)$ and $F(0,-4)$

   (c) $V(0,0)$ focus of the x-axis and passing through point $(8,8)$.

6. Solve the following systems of equations graphically and check by algebraic methods.

   (a) $y^2 = 16x$
   $4x - y = 8$

   (b) $x = y^2 - 3y - 4$
   $x - 3y + 9 = 0$

Your Job and Mathematics

Block and Ross Metal Shop
We Make Anything of Metal
Railway Avenue
Crosstown

What does a machinist do with mathematics? The problems listed here are problems given to the company to solve in the course of its work.

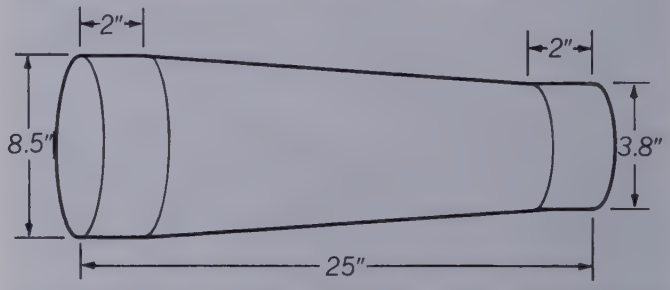
1. Due to the different hardness and brittleness of metals, each metal has a different recommended cutting speed. The number of revolutions per minute of a metal lathe is given by

$$RPM = \frac{12Cs}{\pi d}$$

   where Cs is the recommended cutting speed in feet/minute of the metal and d is the diameter in inches of the metal shaft being *turned*.
   Calculate the recommended *RPM* of a shaft 3.2 inches in diameter if the recommended cutting speed is 136 feet/minute.
   Can you figure out how this formula was developed?

2. An oil contractor wants a cylindrical oil storage tank that will hold 1500 gallons. To allow for expansion he uses the approximation of 7.4 gallons per cubic foot.
   Design what you would consider to be a suitable tank. What would be a suitable height to diameter ratio?

3. A machinist is required to produce the hot air ducts necessary for heating buildings. The one needed is a reducing pipe to fit an (8.5 ± 0.1) inch diameter pipe to a (3.8 ± 0.1) inch diameter pipe. The reduction pipe is to be (25 ± 1) inches. It is to have a sleeve at each end of 2 inches.



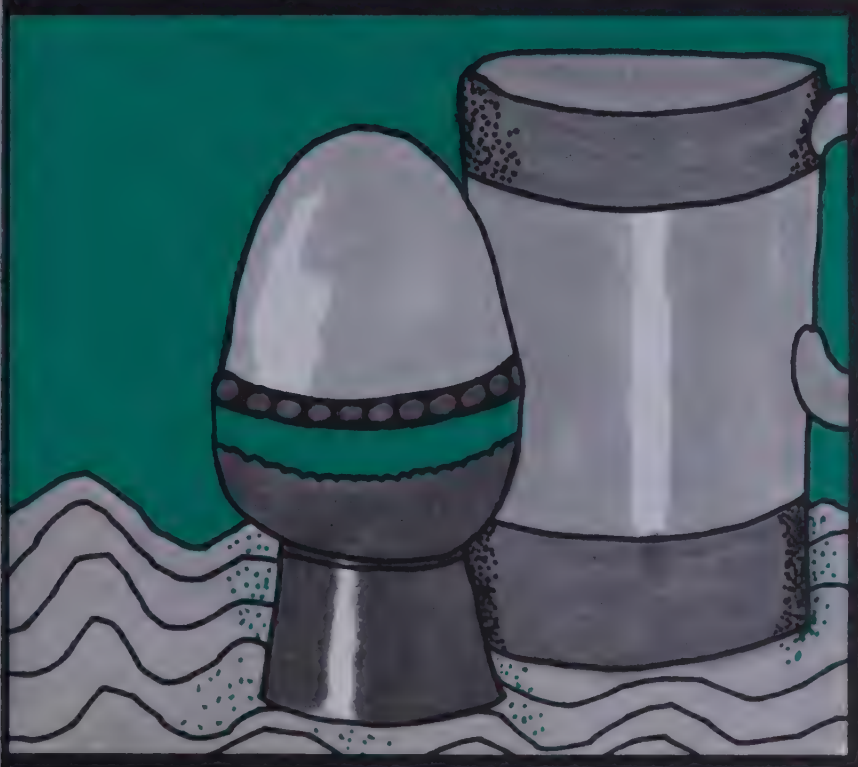
   How would you go about making a pattern for the reducing cone?

   Hint! You may need to make separate pieces and put them together by rivets or soldering. Show how this would be done.

   Compare your solution with your classmates. Is there a "metal" student in your class? If not, enquire of your metal shop teacher if necessary as to the most efficient way to do this.

Chapter Eleven

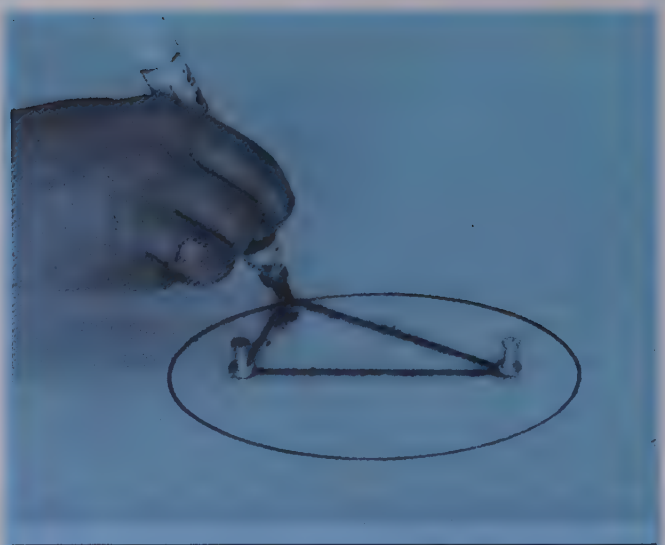
ELLIPSES AND HYPERBOLAS



11·1 Introduction

In the first part of this chapter we will deal with the ellipse. Then, the second part will cover the hyperbola.

We will begin each part with constructions of the curves to familiarize ourselves with the shapes the curves take. The methods of the constructions will vary considerably from each other even though the law for each curve will not. Then, we will follow up with algebraic interpretations of the geometric constructions and study the various relations that exist within each curve.



11·2 Constructions of Ellipses

Projects

Project 1

Purpose

To construct an elliptical curve using string and a pencil.

Method

1. Obtain a string about 10 to 12 inches long, a pencil, two thumbtacks and a tack board on which to pin down a sheet.
2. Tie the string into a loop. On a sheet of paper draw a straight line dividing the sheet approximately in two equal rectangles.
3. Lay the sheet on a tack board and stretch the string loop along the mid-line with half the string lying above the line and half of it lying below the line. Let the string be evenly spaced from each side of the sheet.
4. About half an inch from each end of the stretched loop push a thumbtack (not all the way in so string can slide freely under flat top) on a point on the mid-line. See Figure 1.

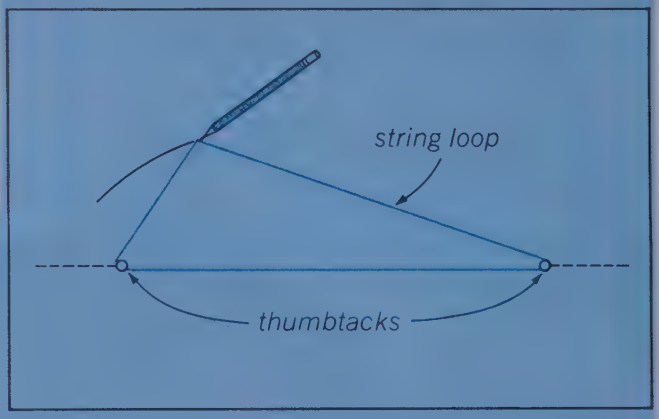


Figure 1

5. With a pencil stretch the loop so that the string forms a triangle, with the two tacks and pencil its vertices.
6. Keeping the string taut with the pencil at all times trace out the curve allowed by the restricting string.
7. Repeat other tracings; for each, alter the distance between the two thumbtacks.

Questions

1. What is the effect on the tracing of the curve when the two tacks are brought closer and closer together?
2. What is the effect on the tracing of the curve when the tacks are separated wider and wider?
3. What are the limits on the positions of the two tacks? What curves occur at these limits?

Project 2

Purpose

To construct an elliptical curve using a pendulum.

Method

1. Make a four- or five-foot pendulum with a flashlight as the bob (light bulb facing downward).
2. Arrange the pendulum high enough so that a time exposure camera can be positioned directly under it to photograph the flashlight when it is in motion.
3. Black out the room for the photographing.
4. Turn the flashlight bulb on. Swing the pendulum to move in an oval orbit. See Figure 1.
5. Open the camera shutter to expose the film to the lit bulb for one complete orbit of the flashlight.

   With a polaroid camera you can view almost immediately the effect produced on the film.

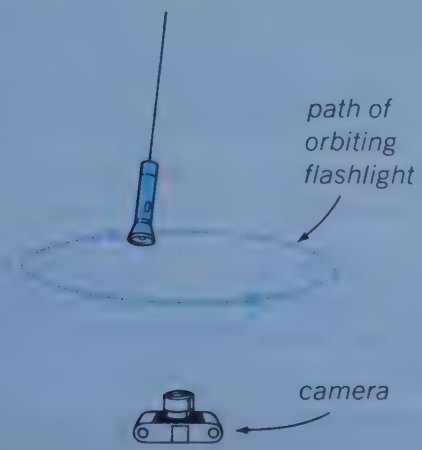


Figure 1

6. If you have a strobe light, use a light colored ball for the bob of the pendulum. Expose the film for one orbit when the strobe light is flashing. This creates a simulated effect of a satellite moving in space.

Questions

1. What are the two limiting orbits that can be photographed by this method?
2. Are there similarities and differences in the resulting curves of Projects 1 and 2?

Project 3

Purpose

To construct an elliptical curve using ratioed line segments and the diameter of a circle.

Method

1. On ruled paper draw a circle, radius 6 centimetres.
2. Through the centre of the circle draw a straight line (diameter) perpendicular to the ruled lines. See Figure 1.
3. Measure each ruled line segment from the diameter to circle. Locate a point on each segment which divides it in a 2:1 ratio from the diameter. For example, in Figure 1, $\overline{AB}$ is divided in two by C so that $\overline{AC}:\overline{BC}$ as 2:1, $\overline{A'B'}$ is divided in two by C' so that $\overline{A'C'}:\overline{B'C'}$ as 2:1.
4. After locating all such points (C) above and below the diameter for each ruled segment, connect them by a smooth flowing curve.
5. Try other similar constructions using varying other ratios (for example, 1 : 1).

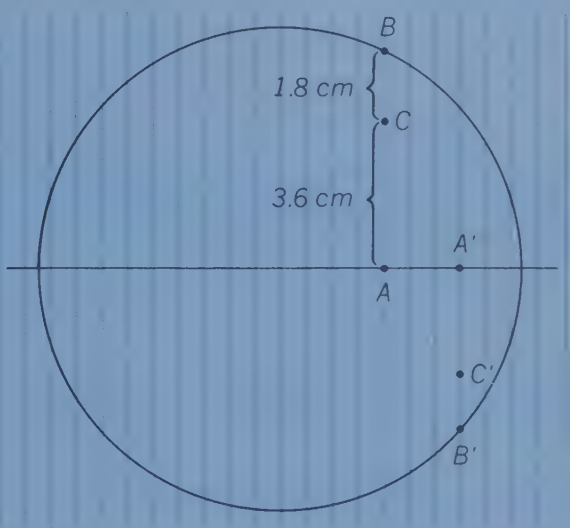


Figure 1

Questions

1. How do the ratios control the shape of the curves?
2. What is the relationship between the ratio and the shape of curve formed?

Project 4

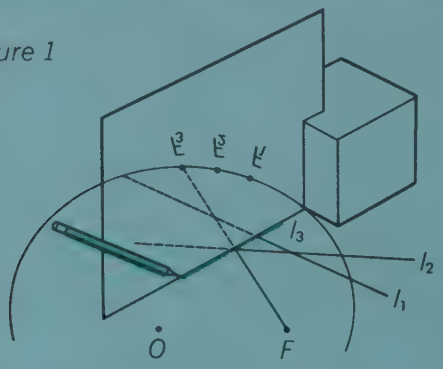
Purpose

To construct an ellipse using the Reflecto.

Method

1. Draw a large circle on a sheet of plain paper.

Figure 1



2. Place a point *F* in the interior of the circle but not at centre *O*.
3. Place Reflecto on paper between point *F* and the circle so that point *F* is reflected onto the circle as shown. (See Figure 1.)
4. Draw a line along Reflecto.
5. Move Reflecto slightly and adjust so image of *F* is on the circle again but in a different position. Draw a line along Reflecto.
6. Continue to move Reflecto so point *F* moves at regular intervals around the circle.
7. What shape does the envelope of lines produce?
8. To create interesting patterns, use the same diagram but place point *F* quite close to *O* (about $\frac{1}{2}$ inch from *O*). Repeat steps 3 to 6. Make the lines quite short.
9. Repeat on separate sheets of paper steps 1 to 7 with:
   (a) *F* nearer centre *O*
   (b) *F* at centre *O*
   (c) *F* nearer circle
   (d) *F* on circle.

   Describe what you observe happens as *F* approaches (or reaches) centre *O* and as *F* approaches (or reaches) the circle.

11·3 Conic Sections

Let us again examine the various curves that are the effect of slicing through a cone. When a cone is sliced at any of the cuts shown in Figure 11-1, the outside edge of each cut produces a curve belonging to the conic section.

(a) circle
(b) ellipse
(c) parabola
(d) hyperbola

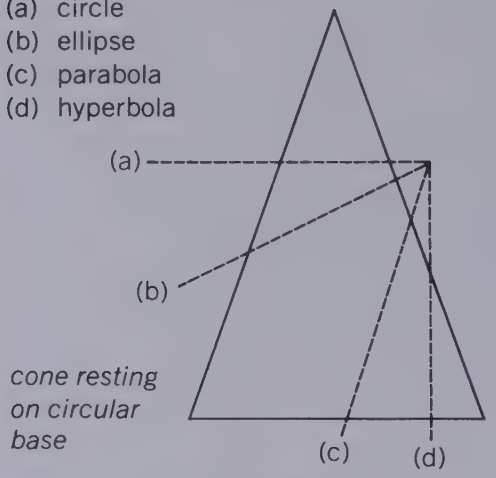
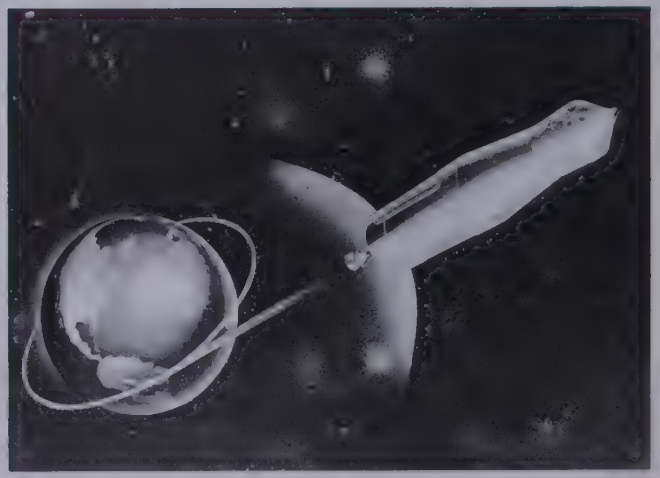


Figure 11-1

If you have a conic section model, take some time to acquaint yourself fully with the curves produced.

An ellipse is a set of points in a plane such that the sum of the distances from every point of the ellipse to two stationary points in the plane is a constant.



The orbits of earth satellites are conic sections.

11·4 Algebraic Treatment

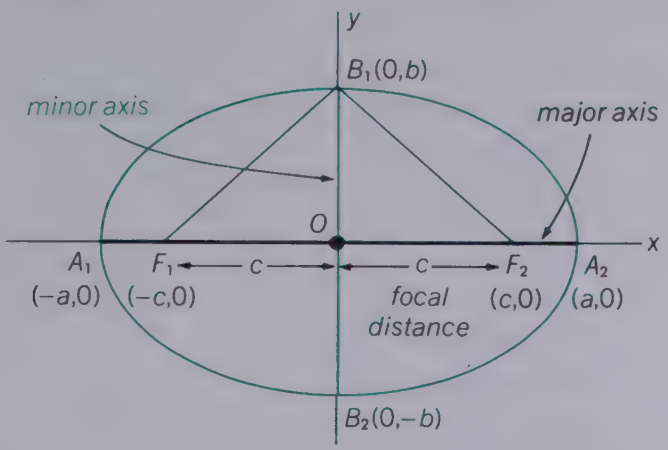


Figure 11-2

In this section we study the ellipse as shown in Figure 11-2. Note the following:

(a) Centre is the origin.

(b) Axes of symmetry are the coordinate axes.

(c) Foci (points) of x-axis, F_1 and F_2.

(d) Major axis A_1A_2; length of major axis $A_1A_2 = 2a$.

(e) Minor axis B_1B_2; length of minor axis $B_1B_2 = 2b$.

(f) Focal distance, $m(\overline{OF_1}) = m(\overline{OF_2}) = c$.

(g) Sum of focal radii = $2a$.

The distance from any point $P(x,y)$ on the ellipse to a focus point is called a *focal radius*. There are always two focal radii in every ellipse.

For the ellipse in Figure 11-2 the defining equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b > 0$$

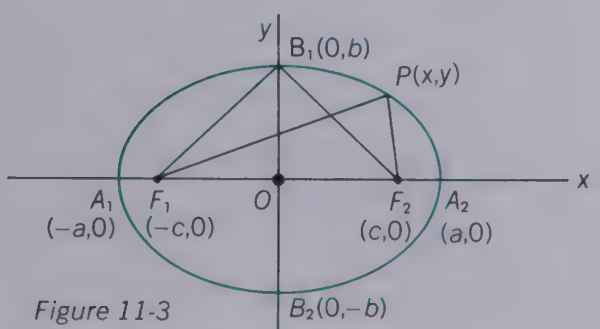


Figure 11-3

For any point $P(x,y)$ on the ellipse (Figure 11-3)

$m(\overline{F_1P}) + m(\overline{F_2P})$ = a constant (by definition)

(i) Consider point P at A_2 on the ellipse.

Then $m(\overline{F_1P}) + m(\overline{F_2P}) = m(\overline{F_1A_2}) + m(\overline{F_2A_2})$

$= (a + c) + (a - c)$ (Why?)

$= 2a.$

That is, the sum of the focal radii = $2a$.

(ii) Consider point P at B_1 on the ellipse.

Then $m(\overline{F_1P}) + m(\overline{F_2P}) = m(\overline{F_1B_1}) + m(\overline{F_2B_1})$

$2a = m(\overline{F_1B_1}) + m(\overline{F_2B_1})$.

But $m(\overline{F_1B_1}) = m(\overline{F_2B_1})$ (Why?)

Note! $\triangle B_1F_1F_2$ is an isosceles triangle.

$\therefore 2a = 2m(\overline{F_1B_1}) = 2m(\overline{F_2B_1})$

and $m(\overline{F_1B_1}) = a$, $m(\overline{F_2B_1}) = a$.

In right triangle B_1OF_2 (Figures 11-3 and 11-4) the measures of the sides are a, b, and c.

$\therefore a^2 = b^2 + c^2$ *(Why?)*

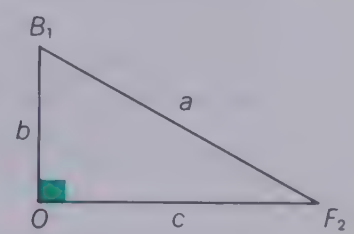


Figure 11-4

Example 1

The following defining equations are for ellipses with foci of the x-axis and centre at the origin.

(a) $\frac{x^2}{100} + \frac{y^2}{36} = 1$ (b) $16x^2 + 25y^2 = 400$

For each of them, find:

(i) the x-intercepts
(ii) the y-intercepts
(iii) the foci
(iv) the length of the major axis
(v) the length of the minor axis
(vi) the focal distance
(vii) the sum of focal radii.

Solution

(a) Compare $\frac{x^2}{100} + \frac{y^2}{36} = 1$

with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$a^2 = 100 = 10^2, a = 10$
$b^2 = 36 = 6^2, b = 6$
$a^2 = b^2 + c^2$
$100 = 36 + c^2$
$c^2 = 64$
$c = 8$

(i) Since $a = 10$,
x-intercepts are $-10, +10$.

(ii) Since $b = 6$,
y-intercepts are $-6, +6$.

(iii) Since $c = 8$,
foci are $(-8, 0)$ and $(8, 0)$.

(iv) Since $a = 10$, then $2a = 20$.
$\therefore$ length of major axis $= 2(10) = 20$.

(v) Since $b = 6$, then $2b = 12$.
$\therefore$ length of minor axis $= 2(6) = 12$.

(vi) Since c is the focal distance then, the focal distance $= 8$.

(vii) The sum of the focal radii is $2a$,
$\therefore$ the sum of focal radii $= 2(10) = 20$.

(b) First rewrite $16x^2 + 25y^2 = 400$ in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Divide through by 400,

$\frac{16x^2}{400} + \frac{25y^2}{400} = \frac{400}{400}$

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$\therefore a^2 = 25 = 5^2, a = 5$
$b^2 = 16 = 4^2, b = 4$

From $a^2 = b^2 + c^2$
$25 = 16 + c^2$
$c^2 = 9$
$c = 3$

(i) x-intercepts are $-5, +5$.

(ii) y-intercepts are $-4, +4$.

Complete (iii) to (vii) as in (a).

Before proceeding do questions 1. and 2 in Exercise 11·4.

Example 2

Find the equations of the ellipses for each of the following (centre of each at origin and foci of x-axis).

(a) $A_1(-5,0), B_1(0,3)$.

(b) $F_2(12,0), A_1(-13,0)$.

(c) Sum of focal radii $= 12$, focal distance $= 3$, foci of the x-axis.

Solution

(a) If $A_1(-5,0)$, then $a = 5$.
If $B_1(0, 3)$, then $b = 3$.
Now make replacements for a and b in

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$\therefore \frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$

or $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is the required equation.

(b) If $F_2(12,0)$, then $c = 12$.
If $A_1(-13, 0)$, then $a = 13$.
From $a^2 = b^2 + c^2$
$13^2 = b^2 + 12^2$
$b^2 = 169 - 144 = 25$
$b = 5$

Now make replacements for a and b in

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\therefore \frac{x^2}{13^2} + \frac{y^2}{5^2} = 1$ is the required equation.

(c) Sum of focal radii $= 2a = 12$, $a = 6$.
Focal distance $= c = 3$

From
$$a^2 = b^2 + c^2$$
$$6^2 = b^2 + 3^2$$
$$b^2 = 36 - 9 = 27$$
$$b = \sqrt{27}$$

Make replacements:

$\therefore \frac{x^2}{36} + \frac{y^2}{27} = 1$ is the required equation.

Example 3

For the ellipse defined by the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \qquad a > b > 0$$

(a) Sketch the graph and label it.

(b) What are the lengths of the major axis and the minor axis?

(c) In what ways does this ellipse differ from one defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

Solution

Note! The equation in this case $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, $a > b > 0$ differs from the one defining the curve in Figure 11-2 in that a^2 is replaced by b^2 and vice versa. Note also that in both cases $a > b > 0$. To find the x-intercepts for $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, let $y = 0$.

Then $\frac{x^2}{b^2} = 1$, $x^2 = b^2$ and $x = \pm b$.

Label the x-intercepts $B_1(-b, 0)$ and $B_2(b, 0)$.

To find the y-intercepts for $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, let $x = 0$.

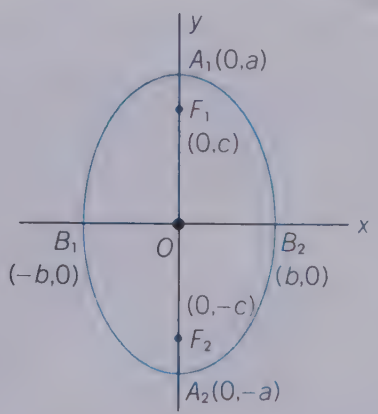
Then $\frac{y^2}{a^2} = 1$, $y^2 = a^2$ and $y = \pm a$.

Label the y-intercepts $A_1(0,a)$ and $A_2(0,-a)$.

Is the egg elliptical in shape?

Then the sketched graph appears as follows:

(a)



(b) The length of the major axis $A_1A_2 = 2a$.
The length of the minor axis $B_1B_2 = 2b$.

(c)

	For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	For $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Major axis	of x-axis	of y-axis
Minor axis	of y-axis	of x-axis
Focal points	of x-axis	of y-axis

Note! For lengths of major and minor axes, focal radius, sum of focal radii, the formulas are the same for both ellipses.

Are the axes of symmetry the same for both ellipses?

Example 4

For each of the defining equations of the ellipses, are the foci of the x-axis or y-axis?

(a) $\frac{x^2}{49} + \frac{y^2}{25} = 1$ (b) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

(c) $9x^2 + 4y^2 = 36$

Solution

First determine whether each equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Refer to Figure 11-2 and Example 3 preceding.

(a) This equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$a^2 = 49$ and $b^2 = 25$ because $a > b > 0$.
By Example 3(c), major axis is of x-axis.
$\therefore$ Foci are of x-axis (because foci are of major axis).

(b) This equation is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

$a^2 = 100$ and $b^2 = 25$ because $a > b > 0$.
By Example 3(c), major axis is of y-axis.
$\therefore$ foci are of y-axis.

(c) $9x^2 + 4y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ $\qquad a^2 = 9$ and $b^2 = 4$

By Example 3(c) major axis is of y-axis.
$\therefore$ foci are of y-axis.

EXERCISE 11·4

1. Define each term (use diagram if necessary): major axis, minor axis, foci of ellipse, sum of focal radii, focal distance, axes of symmetry.

2. For each defining equation of ellipse:

   (a) $\frac{x^2}{4} + \frac{y^2}{5} = 1$ (b) $\frac{x^2}{15^2} + \frac{y^2}{12^2} = 1$

   (c) $\frac{x^2}{144} + \frac{y^2}{169} = 1$ (d) $64x^2 + 100y^2 = 6400$

   determine: (i) the focal points,
   (ii) the length of the minor axis,
   (iii) the sum of the focal radii.

3. Write the defining equations of the ellipses for:

   (a) Focal distance = 12, length of major axis = 30 and foci of y-axis.

   (b) Sum of focal distances = 10, length of minor axis = 8, foci of x-axis.

   (c) $F_1(0,12)$, length of minor axis = 10.

   (d) x-intercept is 20 and y-intercept is 12.

   (e) $F_2(6,0)$ and sum of focal radii = 20.

   (f) $F_1(0,2)$ and x-intercept is 3.

4. (a) What is the effect on the ellipse when F_1 and F_2 are brought close together?

   (b) What is the shape of the ellipse defined by $\frac{x^2}{36} + \frac{y^2}{36} = 1$?

   (c) Give a brief description of the ellipse defined by $\frac{x^2}{1} + \frac{y^2}{10^6} = 1$.

5. In what two straight lines will $\frac{x^2}{4^2} + \frac{y^2}{7^2} = 1$ and $\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1$ be reflections of each other?

   (a) What is the point of intersection of these lines? Where is this point with respect to the two ellipses?

   (b) In how many points do the above ellipses intersect? Do these points lie on the above mentioned straight lines?

6. What are the coordinates of the foci of the ellipse defined by $\frac{x^2}{25} + \frac{y^2}{169} = 1$?

7. Determine the focal points of the ellipse $36x^2 + 100y^2 = 3600$.

8. Which of the following are points of the ellipse $\frac{x^2}{12} + \frac{y^2}{18} = 1$?

   $A(2,2\sqrt{3})$; $B(-3,-2)$; $C(\sqrt{6},-3)$;

   $D(-1,4)$; $E(-2,2\sqrt{3})$; $F\left(\frac{\sqrt{33}}{3},-5\right)$.

9. What is the equation of the ellipse with centre the origin if:

   (a) $F_2(12,0)$ and $B_1(0,5)$

   (b) $F_1(0,9)$ and $A_2(0,-15)$

11·5 Hyperbolas

A hyperbola is a set of points in a plane such that the absolute value of the difference of the distances of each point from two stationary points is a constant.

Geometric Construction

In this section we will follow a geometric construction based on the above definition. We must keep in mind that the absolute value of the differences of the two distances is a constant.

Note! Because there are points P on the curve such that $m(\overline{PF}_1) < m(\overline{PF}_2)$ then $m(\overline{PF}_1) - m(\overline{PF}_2) < 0$. This would not follow the definition that the difference equals a constant. Therefore, it is necessary to take the absolute value of the differences.

$|m(\overline{PF}_1) - m(\overline{PF}_2)| = \text{constant}$

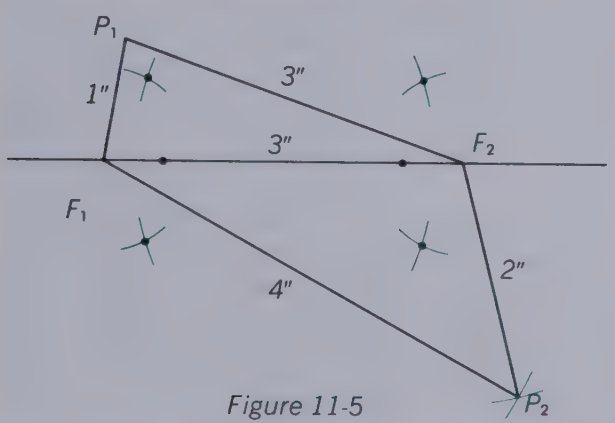


Figure 11-5

The stationary points in Figure 11-5 are foci, the distances PF_1 and PF_2 are called the focal radii. P may be located in many positions, P_1 and P_2 are two of those positions.

The constant in Figure 11-5 for
$|m(\overline{PF}_1) - m(\overline{PF}_2)| = \text{constant}$
is 2 inches. Check this from the diagram.

Locating other positions for P (using the relations of Figure 11-5) results in the diagram of Figure 11-6. The series of points form the appearance of the curve called the *hyperbola*.

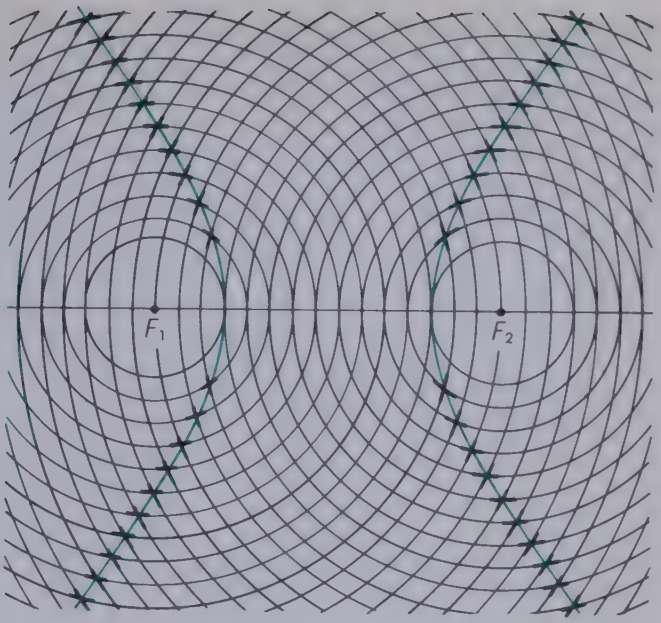


Figure 11-6

Try to reproduce Figure 11-6 on a large sheet of paper.

Hyperbolic curves can be effective in architecture.

Projects

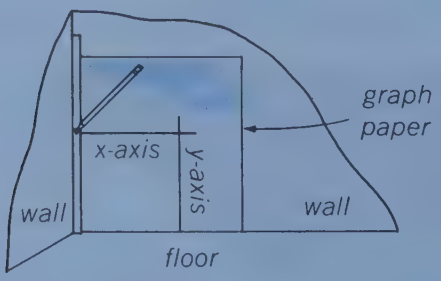
Project 1

Purpose

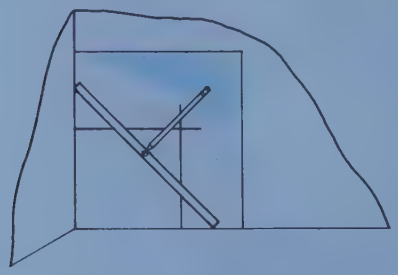
To draw and study hyperbolas of apparently another type.

Method

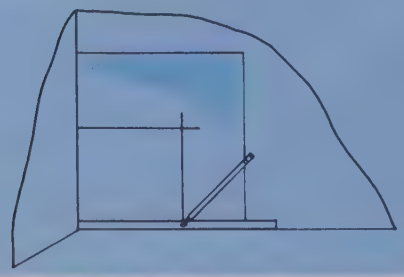
1. Locate a corner and tape a sheet of graph paper on one wall in the corner as shown.
2. Obtain a slat 3 centimetres wide and 1 metre long. Place a hole for your pencil in the middle.
3. Place the slat upright in corner with pencil through hole so as to mark on the paper.
4. Holding the slat and pencil against the paper move the lower end of the slat along the floor until it lies on the floor. Keep the upper end of the slat against the second wall.



(a)



(b)



(c)

5. The shape you have traced is a hyperbola. At each end of the locus you have traced place an axis as indicated.
6. Repeat steps 2 to 5 using the same sheet of graph paper, for (a) a slat 72 centimetres long
   (b) a slat 50 centimetres long.
7. What do you notice about the coordinates of the points of the middle position of the curve? Can you write an equation for each?
8. Use a new piece of graph paper. This time place the pencil hole in the 1 metre long slat:
   (a) 10 cm from the lower end
   (b) 20 cm from the lower end
   (c) 40 cm from the lower end
   (d) 80 cm from the lower end.

   Repeat steps 3 to 5 and 7 for each.

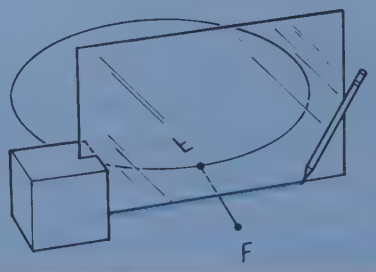
Project 2

Purpose

To draw two hyperbolas using Reflecto.

Method

1. Draw a circle of about 2 inches in diameter.
2. Place a point F in the exterior of the circle and about $\frac{1}{2}$ inch from the circle.
3. Place your Reflecto between the point F and the circle so that point F is mapped onto the circle.
4. Draw a line in full length of Reflecto.
5. Move the Reflecto slightly and repeat steps 3 and 4.
6. Repeat step 5 many times.
7. The envelope of lines describes a hyperbola.
8. Place F on the opposite side of the circle and repeat.



11·6 Algebraic Treatment

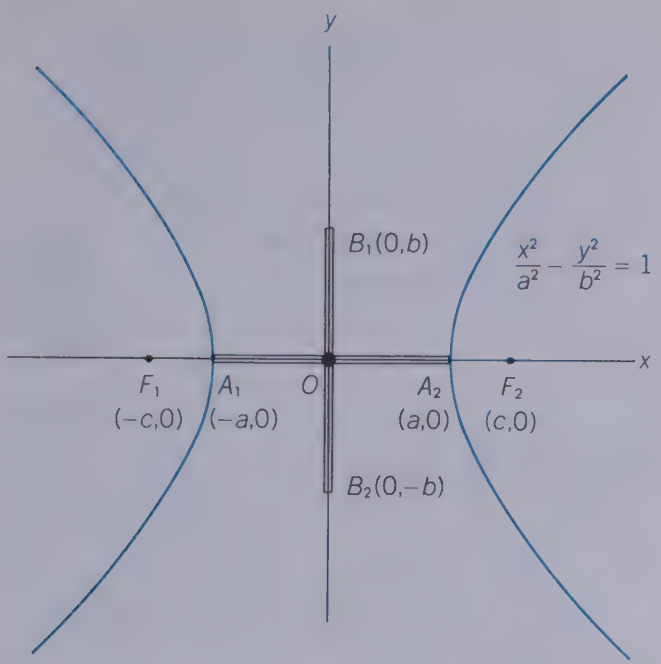


Figure 11-7

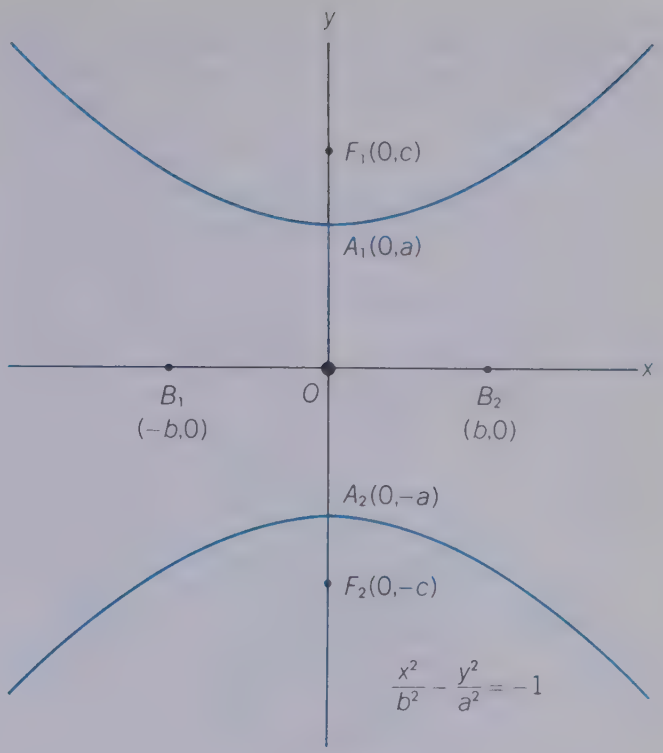


Figure 11-8

The defining equation of the hyperbola in Figure 11-7 is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a > 0,\ b > 0$$

Note the following facts about this hyperbola.

(i) Centre of hyperbola is the origin.

(ii) Axes of symmetry, the coordinate axes.

(iii) Focal points, F_1 and F_2, are of x-axis.

(iv) $\overline{A_1A_2}$ on x-axis is called the transverse axis; length of transverse axis = $2a$.

(v) $\overline{B_1B_2}$ on y-axis is called the conjugate axis; length of conjugate axis = $2b$.

(vi) Focal distance, $OF_1 = OF_2 = c$.

(vii) The x-intercepts are $-a$ and a.

The defining equation of the hyperbola in Figure 11-8 is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1,\ a > 0,\ b > 0$.

Note the following facts about this hyperbola.

(i) Centre of hyperbola is the origin.

(ii) Axes of symmetry, the coordinate axes.

(iii) Focal points, F_1 and F_2 are of y-axis.

(iv) $\overline{A_1A_2}$ on y-axis is called the transverse axis; length of transverse axis = $2a$.

(v) $\overline{B_1B_2}$ on x-axis is called the conjugate axis; length of conjugate axis = $2b$.

(vi) Focal distance, $m(\overline{OF_1}) = m(\overline{OF_2}) = c$.

(vii) The y-intercepts are a and $-a$.

Did you notice the similar properties of the two hyperbolas? In which properties are the two hyperbolas different? In what ways are these properties different?

Note! In the ellipse, $a > b > 0$.
In the hyperbola $a > 0$ and $b > 0$.
There is no major axis nor a minor axis in a hyperbola, because a may be greater than b or b may be greater than a.

The relationship among a, b, and c of a hyperbola is illustrated in Figure 11-9. Note the position and the use of measurements a and b for the rectangle for each hyperbola.

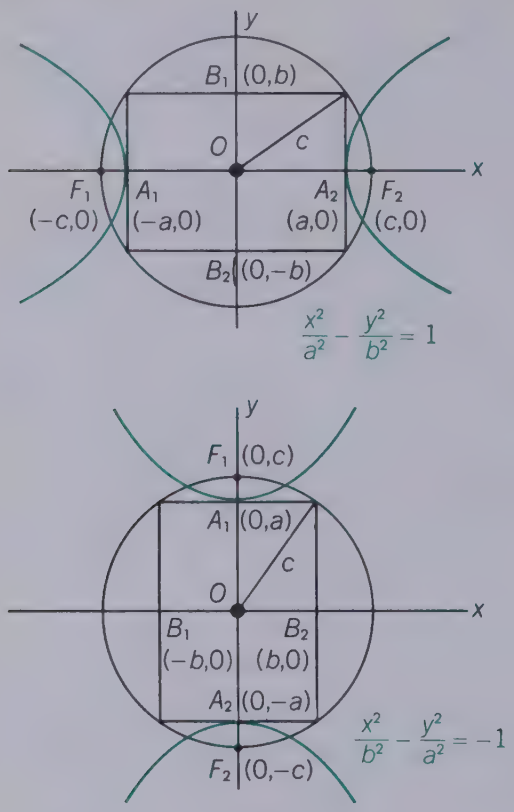


Figure 11-9

Each rectangle is $2a$ units in length and $2b$ units in width. The circle with centre at the origin, and radius c passes through the four vertices of each rectangle and also locates the foci for each hyperbola.

If we lift the two right triangles from each illustration in Figure 11-9 we can then make the following observation in Figure 11-10.

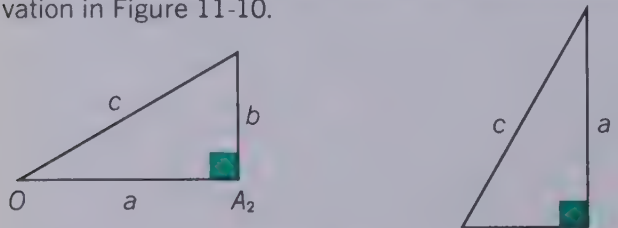


Figure 11-10

$c^2 = a^2 + b^2$ *(Why?)*

From this we may conclude that $c > a$ and $c > b$.

Example 1

The following are defining equations for hyperbolas with foci of the x-axis.

(a) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (b) $36x^2 - 64y^2 = 2304$

For each of them, find:

(i) the x-intercepts,
(ii) the length of the transverse axis,
(iii) the length of the conjugate axis,
(iv) the focal distance,
(v) the foci.

Solution

(a) Compare $\frac{x^2}{9} - \frac{y^2}{16} = 1$

with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$\therefore a^2 = 9 = 3^2, a = 3$

$b^2 = 16 = 4^2, b = 4$

From $c^2 = a^2 + b^2$

$c^2 = 9 + 16 = 25$

$c = 5$

(i) x-intercepts are $-a$ and a, i.e., $-3, +3$
(ii) length of transverse axis $= 2a = 2(3) = 6$
(iii) length of conjugate axis $= 2b = 2(4) = 8$
(iv) focal distance $= c = 5$
(v) foci are $(-c,0)$ and $(c,0)$, i.e., $(-5,0)$ and $(5,0)$.

(b) First rewrite $36x^2 - 64y^2 = 2304$ in the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Divide through by 2304

$\therefore \frac{36x^2}{2304} - \frac{64y^2}{2304} = \frac{2304}{2304}$

$\frac{x^2}{64} - \frac{y^2}{36} = 1$

Compare this with

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$\therefore a^2 = 64, a = 8$

$b^2 = 36, b = 6$

From $c^2 = a^2 + b^2$

$c^2 = 64 + 36 = 100$

$c = 10$

(i) x-intercepts are $-a$ and a, i.e., $-8, +8$

(ii) length of transverse axis $= 2a = 2(8) = 16$

(iii) length of conjugate axis $= 2b = 2(6) = 12$

(iv) the focal distance is $= c = 10$

(v) the foci are $(-c,0)$ and $(c,0)$, i.e., $(-10,0)$ and $(10,0)$.

Before proceeding do questions 1 and 2 in Exercise 11·6.

Example 2

Find the equations of the hyperbolas for each of the following (centre of each at origin):

(a) $A_1(-7,0), B_1(0,4)$ (b) $F_2(8,0), B_2(0,-5)$

(c) Length of transverse axis $= 18$, focal distance $= 10$, conjugate axis of y-axis.

Solution

(a) $a = 7, b = 4$ (Why?)

$\therefore \frac{x^2}{7^2} - \frac{y^2}{4^2} = 1$ or $\frac{x^2}{49} - \frac{y^2}{16} = 1$

is the required equation.

(b) $c = 8, b = 5$ (Why?)

From $c^2 = a^2 + b^2$

$64 = a^2 + 25, a^2 = 39$

$\therefore \frac{x^2}{39} - \frac{y^2}{25} = 1$ is the required equation.

(c) $2a = 18,$

$a = 9,$

$c = 10$ (*Why?*)

From $c^2 = a^2 + b^2$

$100 = 81 + b^2$

$b^2 = 19$

$\therefore \frac{x^2}{81} - \frac{y^2}{19} = 1$ is the required equation.

Example 3

Sketch the graphs of the hyperbolas defined by

(i) $\frac{x^2}{4} - \frac{y^2}{25} = 1$ (ii) $\frac{x^2}{16} - \frac{y^2}{9} = -1$

Solution

To sketch a hyperbola:

From the equation find the measures of a and b.

On the appropriate axes mark the points A_1, A_2, B_1, B_2.

Draw the rectangle as shown in Figure 11-9.

Draw diagonals for the rectangle and extend them in both directions.

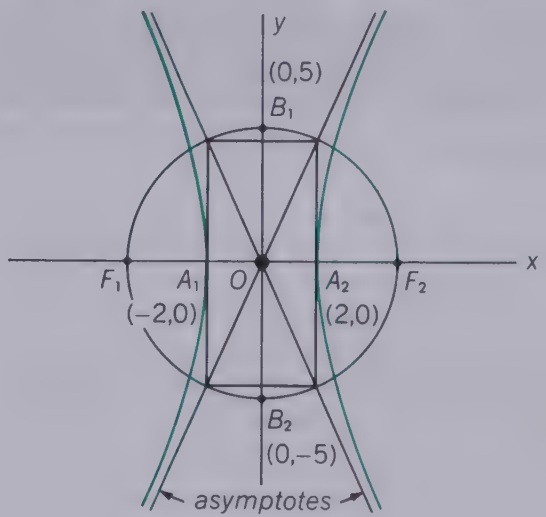
Use compasses to locate foci as shown in Figure 11-9.

Sketch in the hyperbolic curve using the extended diagonals as guides (see note on asymptotes).

(i) $a^2 = 4$ $b^2 = 25$

$a = 2$ $b = 5$

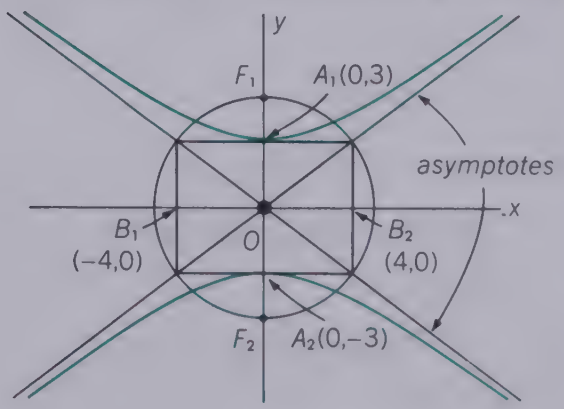
Transverse axis is of x-axis. (See Figure 11-7.)



(ii) $a^2 = 9$ $b^2 = 16$

$a = 3$ $b = 4$

Transverse axis is of y-axis. (See Figure 11-8.)



Curves of conic sections may be observed in many places like the trajectories of the sparks above.

Note! Asymptotes are extensions (unlimited) of the diagonals of the rectangles determined by measures a and b (see Figure 11-9). In further studies of the hyperbola (not covered in this book) the hyperbolic curve comes closer and closer to each asymptote as it moves away from the origin, but it never touches them.

Example 4

For each of the defining equations of the hyperbolas, are the foci of the x-axis or y-axis?

(a) $\frac{x^2}{12} - \frac{y^2}{15} = 1$ (b) $\frac{x^2}{17} - \frac{y^2}{3} = -1$

(c) $3x^2 - 4y^2 + 12 = 0$ (d) $5x^2 - 8y^2 = 40$

Solution

Compare each equation with

(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or (ii) $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$.

(a) $\frac{x^2}{12} - \frac{y^2}{15} = 1$ compares with (i). *Why?*
$\therefore$ foci are of x-axis

(b) $\frac{x^2}{17} - \frac{y^2}{3} = -1$ compares with (ii). *Why?*
$\therefore$ foci of y-axis

(c) Rewrite the equation to compare with (i) or (ii).
$3x^2 - 4y^2 + 12 = 0$
$3x^2 - 4y^2 = -12$ (What operation must be performed to obtain next step?)

$\frac{x^2}{4} - \frac{y^2}{3} = -1$

But this compares with (ii). *Why?*
$\therefore$ foci are of y-axis.

(d) $5x^2 - 8y^2 = 40$

$\frac{x^2}{8} - \frac{y^2}{5} = 1$

This compares with (i). *Why?*
$\therefore$ foci are of x-axis

EXERCISE 11·6

1. Sketch a diagram of a hyperbola and label these parts:

   conjugate axis — focal distance
   transverse axis — axes of symmetry
   centre of hyperbola — several focal radii
   focal points

   What do the a, b and c represent?

2. For each of the following defining equations of hyperbolas:

   (a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (b) $\frac{x^2}{64} - \frac{y^2}{36} = -1$

   (c) $\frac{x^2}{25} - \frac{y^2}{144} = -1$ (d) $9x^2 - 4y^2 = 36$

   determine
   (i) the intercepts (x or y),
   (ii) the foci,
   (iii) the length of the conjugate axis,
   (iv) the length of the transverse axis,
   (v) the focal distance.

3. Write equations of the hyperbolas for each of the following (centre at the origin):
   (a) $A_2(5,0)$, $B_1(0,8)$ (b) $F_1(0,5)$, $B_2(4,0)$
   (c) Length of conjugate axis = 14, one y-intercept at $(0,-4)$.
   (d) Focal distance = 13, length of transverse axis = 24, no y-intercepts.
   (e) x-intercepts -3, $+3$; length of conjugate axis = 16.

4. Sketch the graphs of the following hyperbolas with the aid of asymptotes.

   (a) $\frac{x^2}{36} - \frac{y^2}{16} = 1$ (b) $\frac{x^2}{9} - \frac{y^2}{49} = -1$

   (c) $16y^2 - 9x^2 = 144$

   Geometrically locate the foci of each hyperbola.

5. (a) On a coordinate axes grid draw the graphs of lines represented by $2x - y = 0$ and $2x + y = 0$.

   (b) Locate the points $P(3,6)$, $Q(-3,6)$ and $R(-3,-6)$. Join $\overline{PQ}$ and $\overline{QR}$. Find point T, such that $PQRT$ forms a rectangle. What are the coordinates of T?

   (c) Using the straight lines of (a) as asymptotes and the rectangles of (b) as a guide to x-and y-intercepts, sketch two hyperbolas, one with x-intercepts and one with y-intercepts.

   (d) Geometrically locate the foci of each hyperbola. Tell how you will do this.

   (e) Write the equations of the two hyperbolas sketched in (c).

Note! The transverse axis of one hyperbola is the conjugate axis of the other hyperbola and conversely.

The two hyperbolas are called *conjugate* hyperbolas. Why do you think they are called *conjugate* hyperbolas? What meaning of conjugate is used here?

6. (a) Sketch the hyperbola defined by

$$\frac{x^2}{16} - \frac{y^2}{16} = 1.$$

   (b) Sketch the hyperbola defined by

$$x^2 - y^2 = -16.$$

   (c) Compare the two sketches of the hyperbolas in (a) and (b) with respect to:
   (i) asymptotes,
   (ii) lengths of transverse and conjugate axes,
   (iii) focal lengths,
   (iv) the rectangles.

Hyperbolas which are defined by $x^2 - y^2 = \pm m$ such as 6(a) and 6(b), are called *rectangular hyperbolas*. In rectangular hyperbolas the transverse axis is congruent to the conjugate axis.

7. (a) Draw the graphs of $xy = 60$ and $xy = -60$.
   (b) Draw the axis of symmetry for the two graphs.
   (c) Draw the rectangle for each graph. What do you notice?
   (d) Compare the two graphs of (a) and (b) with respect to:
   (i) asymptotes, (ii) axes of symmetry.
   (e) Are these rectangular hyperbolas? Why?

11·7 $|m(\overline{PF_1}) - m(\overline{PF_2})|$ = Constant

For every point $P(x,y)$ of the hyperbola the absolute value of the difference of the distances of P from the foci is a constant.

> The distance from point P to any focal point is called the focal radius.

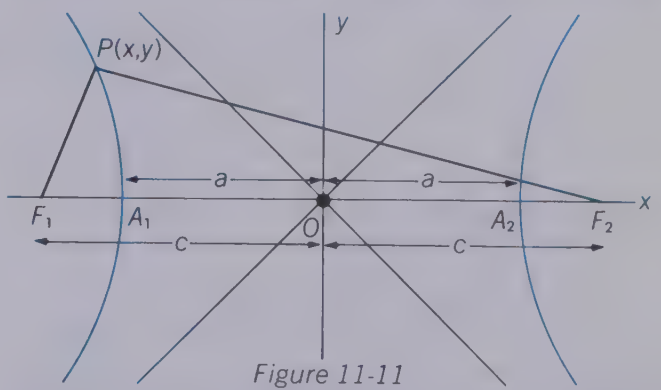


Figure 11-11

If $P(x,y)$ is at A_1, then

$$\begin{aligned} |\text{difference of focal radii}| &= |m(\overline{PF_1}) - m(\overline{PF_2})| \\ &= |m(\overline{A_1F_1}) - m(\overline{A_1F_2})| \\ &= |(c-a) - (c+a)| \\ &= |c - a - c - a| \\ &= |-2a| \\ &= 2a \end{aligned}$$

If $P(x,y)$ is at A_2, then

$$\begin{aligned} |\text{difference of focal radii}| &= |m(\overline{PF_1}) - m(\overline{PF_2})| \\ &= |m(\overline{A_2F_1}) - m(\overline{A_2F_2})| \\ &= |(c+a) - (c-a)| \\ &= |c + a - c + a| \\ &= |2a| \\ &= 2a \end{aligned}$$

Example 1

Find the | difference of focal radii | for the hyperbolas defined by:

(a) $25x^2 - 100y^2 = 2500$ (b) $\frac{x^2}{144} - \frac{y^2}{64} = -1$.

Solution

(a) Rewrite $25x^2 - 100y^2 = 2500$ so that it is possible to determine the value of a.

$$25x^2 - 100y^2 = 2500$$
$$\frac{25x^2}{2500} - \frac{100y^2}{2500} = \frac{2500}{2500}$$
$$\frac{x^2}{100} - \frac{y^2}{25} = 1$$
$$\therefore a^2 = 100 \text{ and } a = 10.$$
$$\text{Hence } |\text{difference of focal radii}| = 2a$$
$$= 2(10)$$
$$= 20$$

(b) For $\frac{x^2}{144} - \frac{y^2}{64} = -1$

$$a^2 = 64 \text{ and } a = 8$$
$$\therefore |\text{difference of focal radii}| = 2a$$
$$= 2(8)$$
$$= 16$$

EXERCISE 11·7

1. Find the |difference of focal radii|, that is $|m(\overline{PF_1}) - m(\overline{PF_2})|$ for the hyperbolas in Exercise 11·6, questions 2 and 4.

11·8 Solution of Linear-Quadratic System of Equations

You will recall that a straight line and a parabola may intersect in two points, one point or no points. We say the solution set has two elements (two points), one element (one point) or no elements (no points).

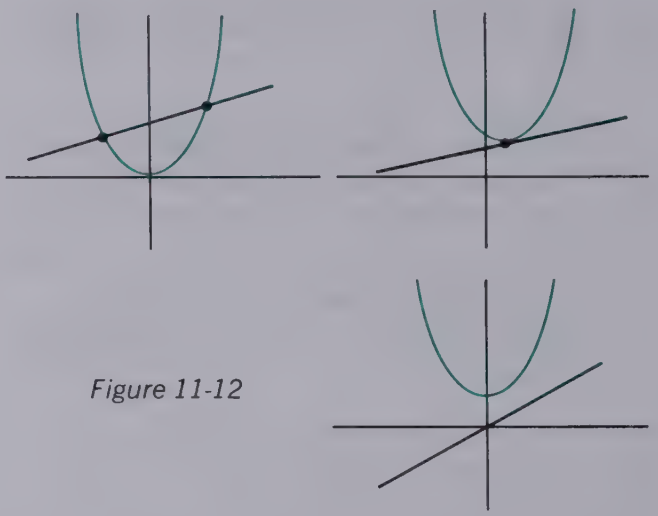


Figure 11-12

Let us consider the possibilities for the solution of

(i) a straight line and an ellipse,

(ii) a straight line and a hyperbola.

After studying the examples and doing several of the questions in the exercise you will be asked these questions:

(a) In how many ways can a straight line intersect a hyperbolic system?

(b) In how many ways can a straight line intersect an ellipse?

Example 1

Solve graphically this pair of linear-quadratic equations.

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$
$$8x - 3y = -24.$$

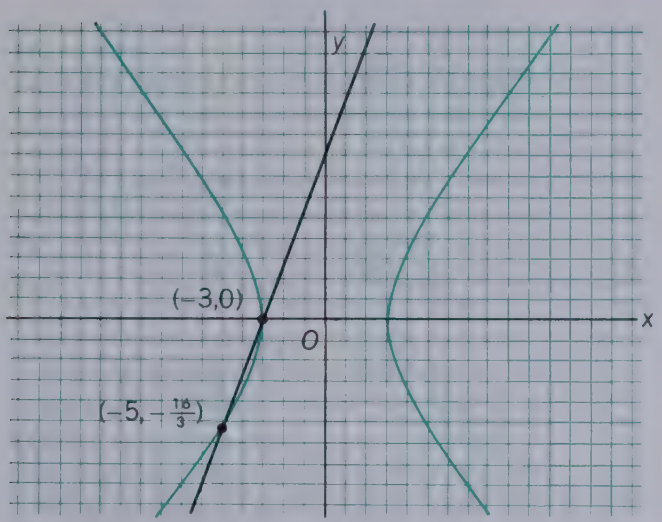
Solution

Table of values for $\frac{x^2}{9} - \frac{y^2}{16} = 1$ or $16x^2 - 9y^2 = 144$

x	3	4	5	6	7	−3	−4	−5	−6	−7
y	0	±3.9	±5.3	±6.9	±8.4	0	±3.9	±5.3	±6.9	±8.4

Table of values for $8x - 3y = -24$

x	6	0	$\frac{3}{2}$	−6
y	24	8	12	−8



From the graph we see there are two points of intersection. These points of intersection are the "solution set" for the linear-quadratic system of equations. The solution set is {(−3,0), (−5,−5.3)}.

Example 2

Solve algebraically:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad ①$$

$$3x - 5y = 15 \qquad ②$$

Solution

Multiply ① by 225 which is the L.C.M. for 25 and 9.

$$9x^2 + 25y^2 = 225 \qquad ③$$

From ②
$$3x = 15 + 5y$$

$$x = \frac{15 + 5y}{3} \qquad ④$$

Is the curve of a bow hyperbolic in shape?

Substitution into ③

$$9\left(\frac{15 + 5y}{3}\right)^2 + 25y^2 = 225$$

$$9\left(\frac{225 + 150y + 25y^2}{9}\right) + 25y^2 = 225$$

$$225 + 150y + 25y^2 + 25y^2 = 225$$

$$50y^2 + 150y = 0$$

$$50y(y + 3) = 0$$

$$50y = 0 \text{ or } y + 3 = 0$$

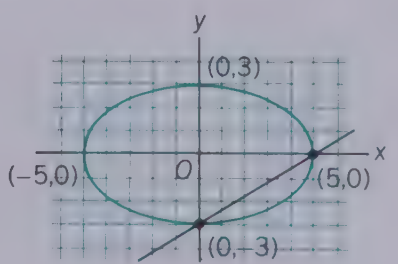
$$y = 0 \text{ or } y = -3$$

From ④
$$x = \frac{15 + 0}{3} \text{ or } x = \frac{15 + 5(-3)}{3}$$

$$= 5 \qquad = 0$$

Solution set = {(5,0), (0,−3)}.

As a check we can sketch the ellipse and the straight line. Does the solution set check?



EXERCISE 11·8

1. What is a linear-quadratic system of equations?

2. Solve the following systems of equations by the graph method.

   (a) $xy = -8$, $x - 2y = 8$

   (b) $\frac{x^2}{9} + \frac{y^2}{64} = 1$, $8x + 3y = -24$

   (c) $\frac{x^2}{9} - \frac{y^2}{9} = 1$, $4x - 5y = 0$

   (d) $\frac{x^2}{225} + \frac{y^2}{100} = 1$, $2x - 3y + 6 = 0$

   (e) $xy = 8$, $x = -y$

   (f) $\frac{x^2}{4} - \frac{y^2}{25} = -1$, $y = 1$

3. Solve the following linear-quadratic systems of equations using algebraic methods. Sketch each set of graphs as a check.

   (a) $xy = 24$, $3y = 2x$

   (b) $\frac{x^2}{9} + \frac{y^2}{25} = 1$, $5x + 3y - 15 = 0$

   (c) $\frac{x^2}{4} - \frac{5y^2}{16} = 1$, $y = 2x - 4$

   (d) $\frac{x^2}{16} - \frac{y^2}{16} = -1$, $3x = 5y - 16$

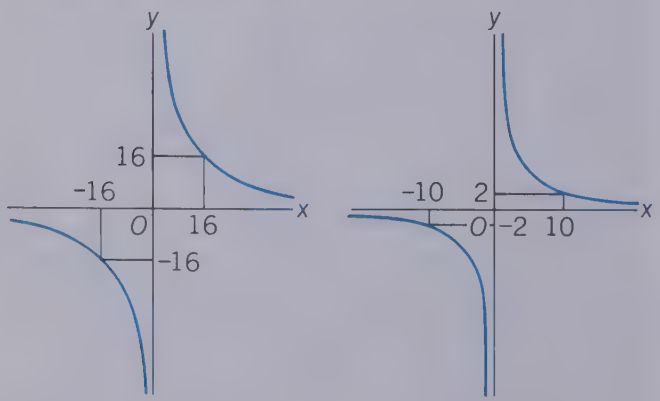
   (e) $x^2 + 2y^2 = 36$, $x + 6 = 2y$

   (f) $\frac{x^2}{16} + \frac{3y^2}{64} = 1$, $3y = 16 - 2x$

   (g) $\frac{x^2}{16} + \frac{y^2}{25} = 1$, $x + y = 10$

   (h) $\frac{x^2}{25} - \frac{y^2}{25} = -1$, $3x - 10y = 15$

4. Can you write equations for each hyperbola?



5. (a) In how many ways can a straight line intersect a hyperbolic system? Discuss.

   (b) In how many ways can a straight line intersect an ellipse? Discuss.

6. Draw two conjugate hyperbolas. Can you draw a straight line that intersects the hyperbolas in 3 or more points? Discuss.

11·9 Applications of Ellipses and Hyperbolas

The ellipse and hyperbola are useful to mathematicians both for their geometric shapes and for their algebraic forms. The problems in this exercise will illustrate both aspects. Hints are given in some problems.

EXERCISE 11·9

1. The force of attraction between two bodies varies inversely as the square of the distance between the centres of the bodies $\left(F = \frac{k}{d^2}\right)$.

   (a) If y represents the force of attraction and x represents the square of the distance, state an equation which represents the relation.

   (b) If the force of attraction is 64 units when the distance between the centres of the bodies is 4 units, calculate the value of the constant k. Write the relation in (a) using this value of k.

   (c) Using the equation in (b) complete this table.

x	1	2	4	6	8	10	12	14	16
y									

   (d) Graph the relation in (c).

   (e) Find the distance between the centres when the force is 1.77 units.

2. The most famous comet is Halley's Comet. It is on a huge elliptical path. It was last seen near the earth in 1910. It appears every 76 years. At present, it is out in the area between Neptune and Pluto. When will it next be near earth?

3. The half-life of the radioactive element Bismuth 53 is 5 days. Draw the Decay Curve for 128 grams of this radioactive element. Does this approximate a hyperbolic curve? What hyperbolic function approximates this curve?

4. Engineers and architects make use of parabolic semi-elliptic and hyperbolic curves in designing many items – for example, buildings, bridges, furniture, appliances. Make a bulletin board display of pictures to illustrate the use of these curves or the solids, paraboloid, ellipsoid, hyperboloid.

5. Boyle's Law states that if the temperature remains constant the volume of a given mass of gas varies inversely as the pressure acting on it – or the product of the mass of gas and the pressure on it is constant. Show this relation graphically.

6. Charle's Law states that, if the pressure remains constant the volume of a given mass of gas varies directly as the absolute temperature. Show this relation graphically. Is this a quadratic relation? Explain.

7. If the volume of a gas remains constant how do you think pressure of the gas and the absolute temperature are related? Do you think this is a quadratic relation? Explain.

Using the Library

1. There are so many mathematicians during this period we can select only a few. Most of the areas of their studies are so advanced that we shall not go into detail on them. Some of these mathematicians had very interesting lives. Continue work on your time line.

   Galois, born – invented the word "group" used in Modern Algebra
   Riemann, born – non Euclidean geometry
   End of war between Canada and the Americans in which Tecumseh was killed
   Dedekind, born – died in 1916. A newspaper in 1899 listed him as having died. He replied to the editor "The day may be right, but the year is certainly wrong."
   Roencan, born – founder of modern topology
   Max Planck, born – mathematician and scientist
   David Hilbert, born – outlined a systematic study of axioms of geometry and algebra
   British North American Act
   Lebesgue, born
   Harold Bohr, born – brother of Niels
   "Last Spike" driven at Craigellachie
   Emmy Noether, born – one of the few women noted as a mathematician
   Birkoff, born – developed, among other things, the theory of coloring of maps
   Louis de Broglie, born

2. Recount the events leading up to Galois' death at the youthful age of 21.

3. Emmy Noether, it is reported, was so kind, humorous and sociable that everybody loved her, yet she was criticized by many who had met her only briefly. Research Emmy Noether, commenting on interesting aspects of her personality.

4. Research the life and times of Einstein. What kind of a man was he? What kind of a life did he lead in the later life? He had two great interests. What were they?

5. Research the reflector property of a hyperbolic shaving mirror. How does it work?

REVIEW EXERCISE 11

1. For each of the ellipses defined by
   (a) $4x^2 + 144y^2 = 576$ (b) $100x^2 + 81y^2 = 8100$
   find:
   (i) the x-intercepts,
   (ii) the y-intercepts,
   (iii) the foci,
   (iv) the length of the major axis,
   (v) the length of the minor axis,
   (vi) the sum of the focal radii.
   Sketch the two ellipses.

2. Find the equation of the ellipse with centre the origin when:
   (i) length of major axis is 13 and one focal point is (5,0),
   (ii) one focal point is (0,−12) and the sum of the focal radii is 30,
   (iii) the two x-intercepts are −7 and +7 and the y-intercepts are −11 and +11.

3. Is (2,10) a point of the ellipse $\frac{x^2}{12} + \frac{y^2}{15} = 1$?

4. For each of the hyperbolas defined by
   (a) $\frac{x^2}{144} - \frac{y^2}{256} = 1$ (b) $100x^2 - 576y^2 = -57{,}600$
   determine:
   (i) the x- or y-intercepts,
   (ii) the foci,
   (iii) the length of the transverse axis,
   (iv) the length of the conjugate axis,
   (v) the focal distance,
   (vi) |the difference of focal radii|.
   Sketch the two hyperbolas with asymptotes as guides.

5. Solve algebraically the following pairs of linear-quadratic equations.
   (a) $\frac{x^2}{100} + \frac{y^2}{12} = 1$ (b) $x^2 - 4y^2 = 36$
   $x + 5y = 20$ $x - 4y + 6 = 0$
   Check your answers graphically.

Your Job and Mathematics

ABC Construction Company
Calgary, Alberta

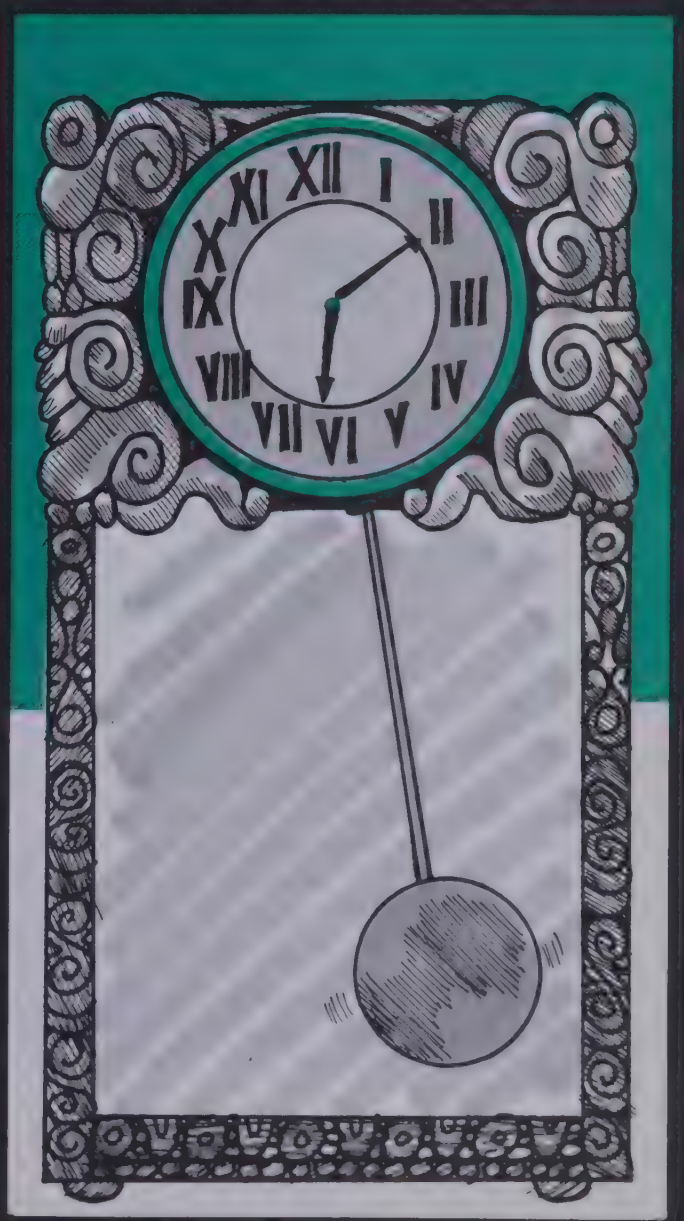
An employee must be able to make simple calculations involved in making bids on jobs. The calculations must be accurate for if the calculations are too small the company could go bankrupt and if they are too large the company will not get the job. The following two problems illustrate the need for mathematics.

1. In preparation of a site (512′ × 760′) for building, one portion of the site (512′ × 312′) must be lowered 22 inches. The other portion of the lot consists 60% of holes from old basements. Each hole is 11 feet deep. Dirt moved is measured in cubic yards. How many additional cubic yards of "fill" will be needed to level the site, after the fill from the higher portion is put in the lower portion?
   The formula used is $y = \frac{LWD}{27}$. Can you interpret this formula?

2. In building the Mica Dam on the Columbia River north of Revelstoke, B.C. engineers are required who use very complex mathematics. To illustrate some of the calculations (a simple case), the force of the water on a dam can be calculated. The force will then be used by the engineers to determine the size and shape of the dam.
   The formula $F = AHW$ is used to calculate the total force in pounds where A is the area (in square feet) of the force of the dam under water, H is the depth of the water (in feet), and W is the weight of water per cubic foot.
   Calculate the force on a dam shown here if water weighs 62.8 pounds per cubic foot.

Chapter Twelve

THE PERIODIC NATURE OF TRIGONOMETRIC FUNCTIONS



12·1 Functions of Quadrantal Angles

A quadrantal angle has a measure of 0° or any multiple of 90°. Thus, when the radius vector coincides with an axis, a quadrantal angle has been generated.

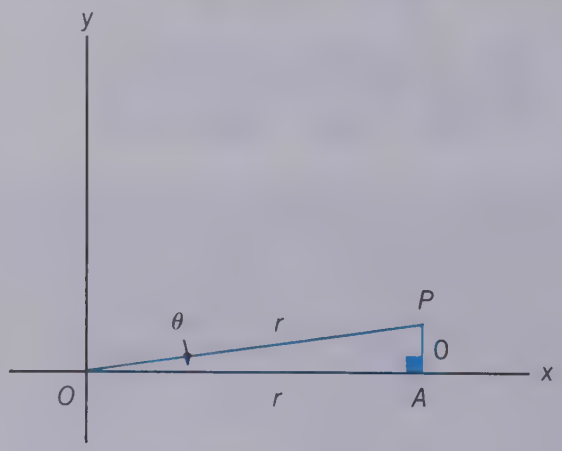


Figure 12-1

Let us consider the angle θ equal to zero. We attempt a sketch (Figure 12-1) showing radius vector OP in a position such that $\triangle OPA$ is formed. The diagram is not a true representation of $\theta = 0°$ but it will help our imaginations to picture the real drawing.

Since θ is 0°, we label $\overline{PA}$ of magnitude zero. $m(\overline{OA})$ will equal the measure of the radius vector r. Then

$$\sin 0° = \frac{0}{r} = 0 \qquad \csc 0° = \frac{r}{0} \text{(undefined)}$$

$$\cos 0° = \frac{r}{r} = 1 \qquad \sec 0° = \frac{r}{r} = 1$$

$$\tan 0° = \frac{0}{r} = 0 \qquad \cot 0° = \frac{r}{0} \text{(undefined)}$$

For csc 0° and cot 0° if we take θ as a very small angle, then $m(\overline{PA})$ can be an extremely small value. Thus r divided by such an extremely small quantity will give csc 0° and cot 0° an extremely large value, so large as to be infinite. We may then reason that as θ approaches 0°, csc 0° and cot 0° approach infinity (∞).

As θ approaches 90° the sketch (Figure 12-2) shows the relationships of the sides. When $\theta = 90°$, $m(\overline{PA}) = r$ and $m(\overline{OA}) = 0$.

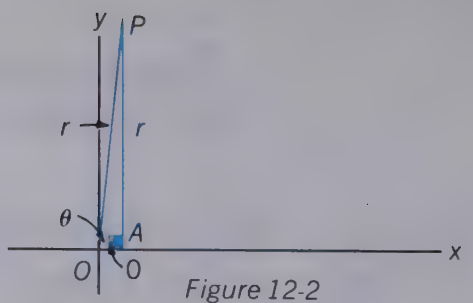


Figure 12-2

Then

$$\sin 90° = \frac{r}{r} = 1 \qquad \csc 90° = \frac{r}{r} = 1$$

$$\cos 90° = \frac{0}{r} = 0 \qquad \sec 90° = \frac{r}{0} \text{(undefined)}$$

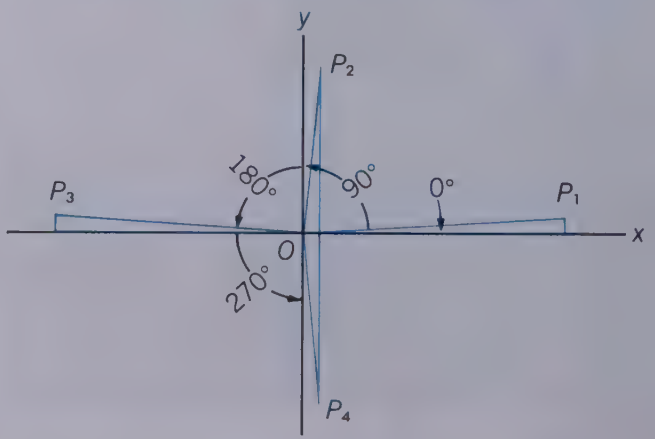
$$\tan 90° = \frac{r}{0} = \text{(undefined)} \qquad \cot 90° = \frac{0}{r} = 0$$

Explain why sec 90° and tan 90° are undefined.

EXERCISE 12·1

1. With the aid of the following diagram or by your own sketch of each quadrantal angle, tabulate the results as outlined.

Angle	sin	cos	tan	csc	sec	cot
0°		0				
90°						
180°					−1	
270°						



12·2 Line Values of Functions

Let us construct a circle with centre the origin and a radius of 1 unit. Sometimes we call it a *unit circle* (Figure 12-3). The points of the circle will then be all the points P of the radius vector as it generates the angle θ.

From Figure 12-3 we read:

$$\sin\theta = \frac{m(\overline{PA})}{1} = m(\overline{PA})$$

$$\cos\theta = \frac{m(\overline{OA})}{1} = m(\overline{OA})$$

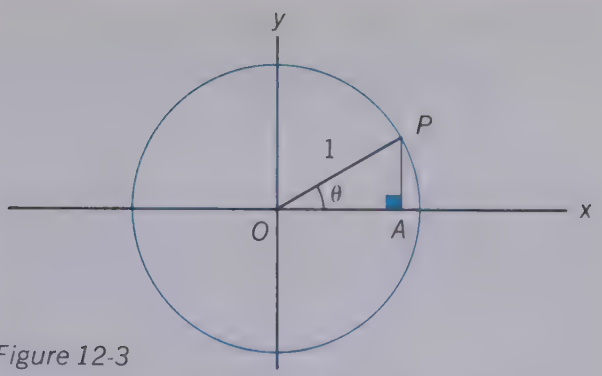


Figure 12-3

With a unit circle such as in Figure 12-3 we can express the sin θ ($\theta \in R$) by the line segment PA. We call $m(\overline{PA})$ the *line value* for such θ. Similarly $m(\overline{OA})$ is the line value of cos θ ($\theta \in R$).

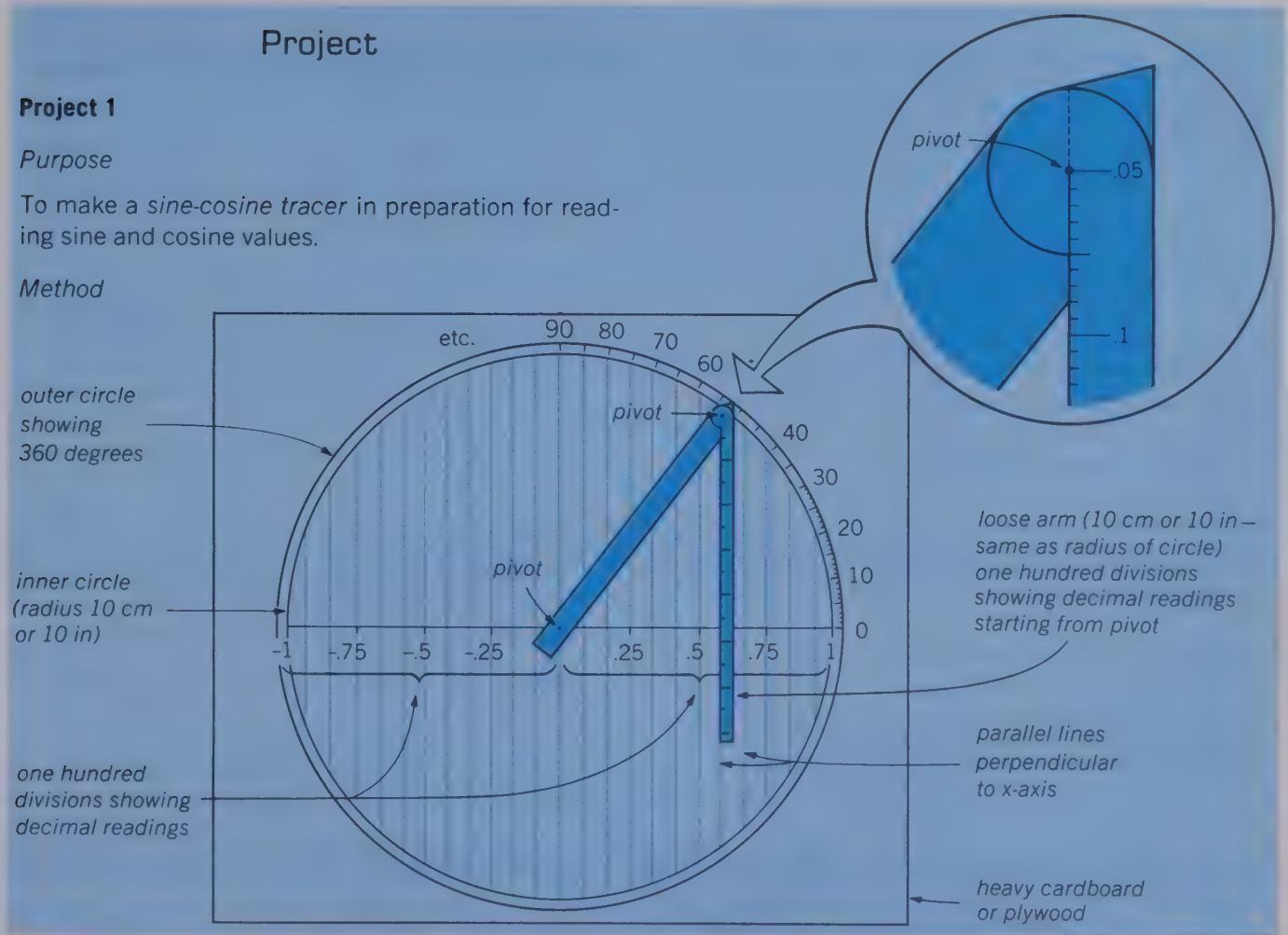
Project

Project 1

Purpose

To make a *sine-cosine tracer* in preparation for reading sine and cosine values.

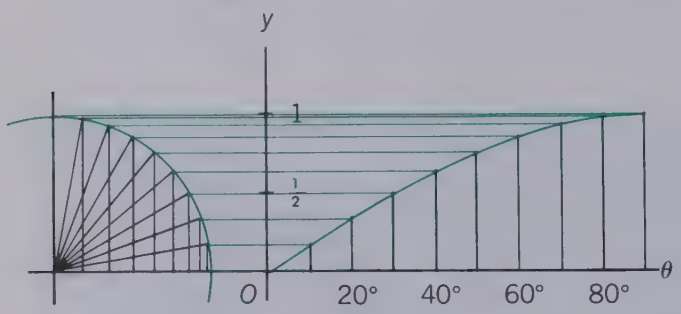
Method



12·3 Periodic Nature of the Functions

We can now take the line values for sin θ and place them on a Cartesian-coordinate grid. The composite graph of all the line values will then represent the nature of the sine function.

Figure 12-4 shows the line values of sin θ for $0° \leq \theta \leq 90°$ in 10 intervals. The heavy vertical lines are the line values for sine θ and they are transferred from the portion of the unit circle to the Cartesian grid.



(a) (b)

Figure 12-4

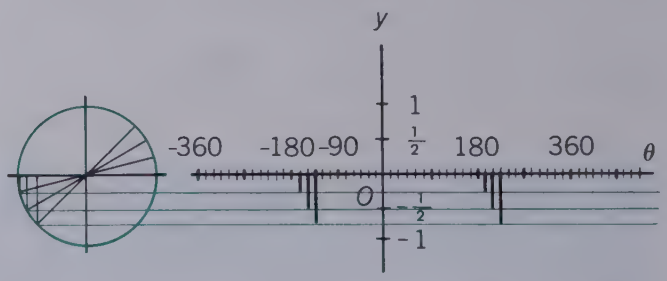
Since θ has an infinite number of values in $0° \leq \theta \leq 90°$, there will be a corresponding infinite number of line values. When all of the line values are graphed in their respective positions, the sine curve as outlined above (Figure 12-4) emerges.

For discussion:

1. Why should the 1 be placed in that position on the y-axis of Figure 12-4 (b)?
2. For what θ is the line value of the graph equal to $\frac{1}{2}$?
3. When θ increases beyond 90° what will be happening to the line values of sin θ?

EXERCISE 12·3

1. On about 4 or 5 sheets of paper taped together to form a long scroll, carefully draw a unit circle (radius about 3 inches) and a grid to the right as shown in this illustration. Divide the circle into 15° angles and draw in the line values for sin θ as in Figure 12-4. Transfer the line values to their respective places on the grid where $-360° \leq \theta \leq 360°$. Draw a smooth curve to represent all the line values of sin θ. This curve is known as the sine wave.



(a) What is the maximum value of the sine curve?

(b) What is the minimum value?

(c) For what θ is the sine function maximum?

(d) For what θ is the sine function minimum?

The greatest displacement of the sine curve from the horizontal axis is called its amplitude.

Since the line values on the grid become y values then y is a function of θ. In this case $y = \sin \theta$. Therefore, the graph you have drawn is the graph of $y = \sin \theta$.

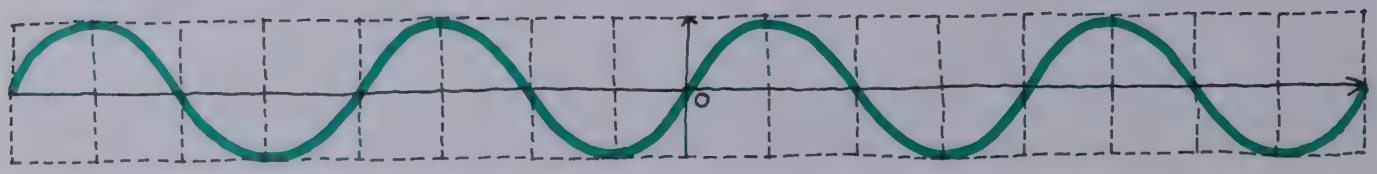
(e) At what values of θ is $y = 0$?

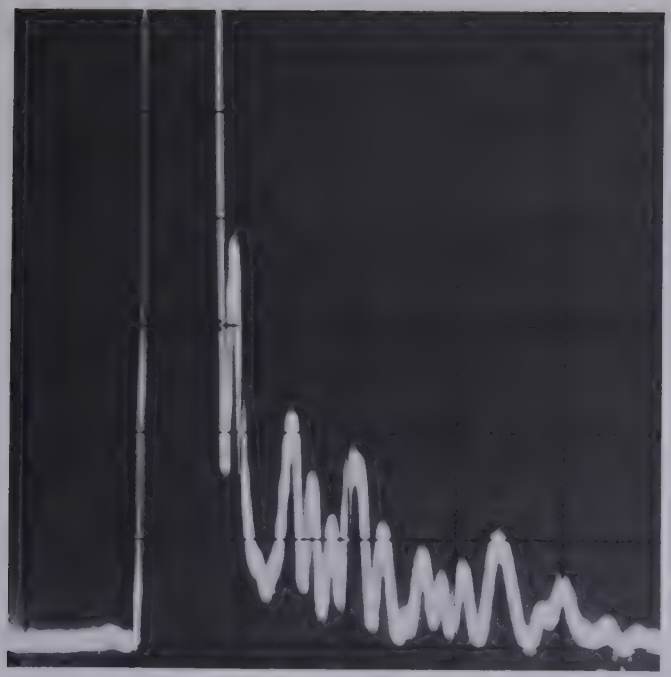
(f) For what values of θ is $y = \frac{1}{2}$? $y = -\frac{1}{2}$? Is this consistent with Section 6·3?

(g) How often does the sine curve repeat itself?

The fact that the curve repeats its motion is an indication that $y = \sin \theta$ is a periodic function.

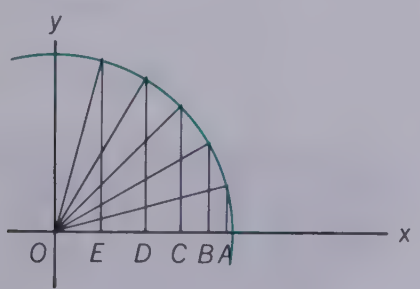
(h) The *period* of $y = \sin \theta$ is ■.





The sine curve has great value in the interpretation of certain physical phenomena. A musical note of one frequency is transmitted by a sound wave shaped like the sine curve. Electromagnetic waves like those of radio transmitters travel in sine waves. In the study of sine wave motion, reference is frequently made to wavelength. Wavelength is the length of the wave for one period.

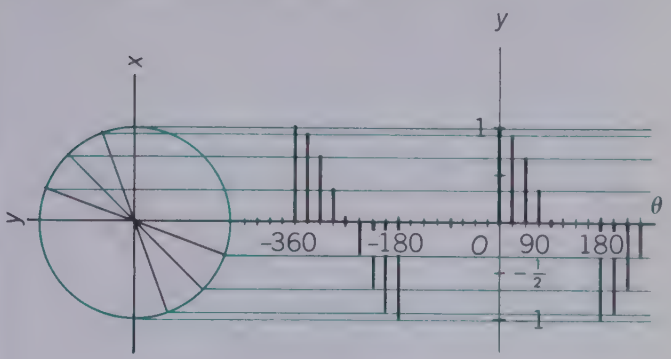
2. On a scroll of same size as in question 1, and with the same scale as in question 1, draw the curve representing all the line values of $y = \cos\theta$ for $-360° \leq \theta \leq 360°$.



Note that the line values for $\cos\theta$ as shown in this illustration are as follows:

$\cos 15° = m(\overline{OA})$
$\cos 30° = m(\overline{OB})$
$\cos 45° = m(\overline{OC})$
$\cos 60° = m(\overline{OD})$
$\cos 75° = m(\overline{OD})$

Therefore, the unit circle relative to the grid must be arranged in this fashion.



Complete the graph.

(a) From the curve, what is the value of y for $\theta = 0°$? For what other θ is y the same value? Is this a maximum or minimum y?

(b) For what θ is the curve a minimum y?

(c) What, then, is the amplitude for the cosine curve?

(d) What is the period of $y = \cos\theta$?

(e) Is the cosine wave similar to the sine wave? Where do the differences in the two curves occur? In what ways are they alike?

3. Take the scroll graph of $y = \cos\theta$ and superimpose it over the scroll graph of $y = \sin\theta$ so that the coordinate axes coincide. Look against a lighted window to see both graphs. (If you have equally marked axes then proceed to the questions following. If your axes were not equally marked, construct graphs of $y = \sin\theta$ and $y = \cos\theta$ on the same coordinate axes.)

(a) Are the graphs of $\sin\theta$ and $\cos\theta$ coincident? How far to the right or left do you have to move the top graph so that the two curves coincide? By what value of θ are they not coincident?

If the graphs had coincided they would have been in *phase*. As you observe them the two graphs are out of phase by 90°.

(b) Which graph has to be moved to the right 90° to be in phase with the other?

(c) Which of the two relations is correct?
$\sin\theta = \cos(\theta + 90°)$ or $\cos\theta = \sin(\theta + 90°)$

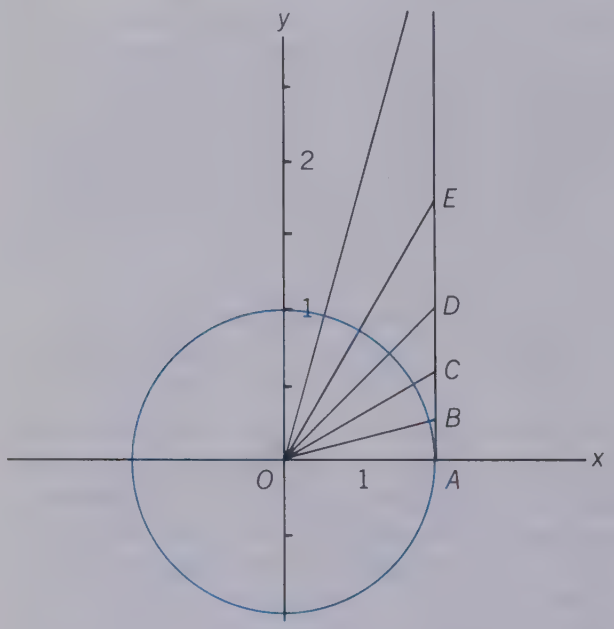
(d) Is $\sin\theta = \cos(\theta - 90°)$?

(e) Is $\cos\theta = \sin(\theta - 180°)$?

(f) Is $\cos\theta = \sin(\theta + 270°)$?

(g) Is $\sin\theta = \cos(\theta + 180°)$?

4. Draw a graph of $y = \tan\theta, -180° \leq \theta \leq 360°$. Since you are concerned with transferring line values to a grid, it is necessary to devise a method of drawing line values for $\tan\theta$. The following sketch should be helpful.



Quadrant I is divided into 15° angles.

$$\tan\angle BOA = \tan 15° = \frac{m(\overline{BA})}{1} = m(\overline{BA})$$

$$\tan\angle COA = \tan 30° = \frac{m(\overline{CA})}{1} = m(\overline{CA})$$

$$\tan\angle DOA = \tan 45° = \frac{m(\overline{DA})}{1} = m(\overline{DA})$$

$$\tan 60° = \frac{m(\overline{EA})}{1} = m(\overline{EA})$$

$m(\overline{BA})$, $m(\overline{CA})$, $m(\overline{DA})$, etc., are the line values for transfer to the grid.

(a) Is the tangent graph a smooth curve (continuous) like the sine curve?

(b) Does a graph value exist for tan 90°? For what other θ does the same situation occur?

(c) Can you read the value for tan 45°? For what other θ are the values equal to tan 45°? For what θ are the values equal to -1?

(d) Is there an amplitude for $y = \tan\theta$? Why?

(e) What is the period for $y = \tan\theta$? Does the period differ from $y = \sin\theta$ and $y = \cos\theta$? How?

(f) In what ways does the tangent curve differ from the sine and cosine curves?

5. For $y = \sin 2\theta$:

Complete the following table with the help of a sine-cosine tracer or trigonometric tables for $0° \leq \theta \leq 360°$.

θ	0°	30°	60°	90°	120°	150°	180°
y	0	1.0000	1.7320	2.0000			

210°	240°	270°	300°	330°	360°

(a) Sketch a graph of $y = \sin\theta$ for $0° \leq \theta \leq 360°$.

(b) From the above table of values, and on the same axes as (a) draw the graph of $y = \sin 2\theta$.

(c) Do the amplitudes of $y = \sin\theta$ and $y = \sin 2\theta$ differ? What are the amplitudes?

(d) What is the period of $y = \sin 2\theta$ when compared with $y = \sin\theta$?

How is the sine curve represented in this photograph?

6. For $y = \cos 2\theta$, $0° \leq \theta \leq 360°$, work out a table of values and sketch its graph. On the same axes sketch $y = \cos \theta$.

   (a) Compare the graphs of $y = \cos \theta$ and $y = \cos 2\theta$. How do they differ? How are they the same?

   (b) Do the amplitude and period differ for the two? If so, how do they differ?

   (c) On the same axes draw the graph of $y = \cos \frac{1}{2}\theta$.
   Compare this graph with the other two.

7. Sketch graphs of $y = \sin 3\theta$, $y = \sin \frac{1}{2}\theta$ and $y = \sin \frac{1}{3}\theta$. Compare them with graphs of $y = \sin \theta$ and $y = \sin 2\theta$.

   (a) Draw a generalization for the graph of $y = \sin(b\theta)$.

   (b) How would you determine the period of $y = \sin(b\theta)$?

   (c) Does the generalization regarding period hold also for $y = \cos(b\theta)$?

8. Refer to the graphs in questions 1 and 2.

   (a) Set up a table of values for $y = 2\sin \theta$ and $y = 2\cos \theta$. Graph each.

   In what way do the graphs of $y = 2\sin \theta$ and $y = 2\cos \theta$ differ from $y = \sin \theta$ and $y = \cos \theta$?

   Set up tables of values for $y = 3\sin \theta$ and $y = \frac{1}{2}\sin \theta$. How do graphs of $y = 3\sin \theta$ and $y = \frac{1}{2}\sin \theta$ differ from $y = \sin \theta$?

   (b) Can you generalize regarding the amplitude for $y = a\sin \theta$, $y = a\cos \theta$?

   (c) For $y = 3\sin 2\theta$, determine the amplitude and the period. Do the same for $y = \frac{1}{2}\cos 3\theta$ and $y = 2\sin \frac{1}{2}\theta$.

   (d) For $y = a\sin(b\theta)$ and $y = a\cos(b\theta)$, tell how to determine the amplitude and the period in terms of a and b.

9. (a) What are the period and amplitude of $y = \sin \theta$?

   (b) What is the period of $y = \sin \frac{1}{4}\theta$?

Does this represent periodic motion?

   (c) What is the amplitude of $y = \frac{5}{2}\sin \theta$?

   (d) What are the period and amplitude of $y = \frac{5}{2}\sin \frac{1}{4}\theta$?

10. (a) What are the period and amplitude of $y = \cos \theta$?

   (b) What is the period of $y = 2\cos \theta$?

   (c) What is the amplitude of $y = \cos \frac{1}{3}\theta$?

   (d) What is the period of $y = \cos \frac{1}{3}\theta$?

   (e) What is the amplitude of $y = 2\cos \theta$?

   (f) What are the period and amplitude of $y = 2\cos \frac{1}{3}\theta$?

11. (a) What is the period of $y = \tan \theta$?

   (b) What is the period of $y = \tan 2\theta$?

   (c) What is the period of $y = \tan \frac{2}{3}\theta$?

Project

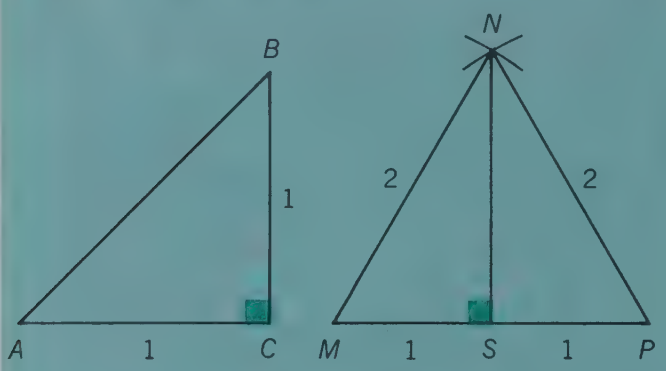
Project 1

Purpose

To learn the relation among sin θ, cos θ, tan θ and cot θ.

Method

1. Draw two triangles, $\triangle ABC$ and $\triangle MNP$, as indicated in the following diagrams:



2. For $\triangle ABC$ determine the measure of side $\overline{AB}$ and the measures of $\angle A$ and $\angle B$.

3. For $\triangle MNS$ determine the measure of side $\overline{NS}$ and the measures of $\angle M$ and $\angle MNS$.

   Check with your teacher regarding the measures in 2 and 3 above before you proceed to 4.

4. From the two triangles, *ABC* and *MNS*, fill out the following table for sin θ, cos θ, tan θ and cot θ when θ is given.

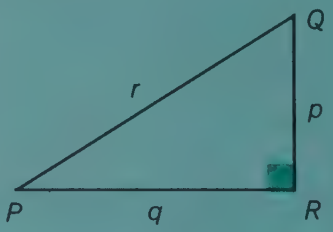
	30°	45°	60°	90°	120°	225°	315°
sin θ							
cos θ							
tan θ							
$\frac{\sin\theta}{\cos\theta}$							
cot θ							
$\frac{\cos\theta}{\sin\theta}$							

5. Under each θ given in the table in question 4, take the values of sin θ and cos θ and compute the ratios $\frac{\sin\theta}{\cos\theta}$ and $\frac{\cos\theta}{\sin\theta}$. Fill these into the appropriate places in the table.

6. (i) Compare the values of tan θ and $\frac{\sin\theta}{\cos\theta}$ for each given θ.

   (ii) Compare the values of cot θ and $\frac{\cos\theta}{\sin\theta}$ for each given θ.

7. What observations do you make from these comparisons? Can you make a general statement for each of the comparisons?

8. 

   For the given triangle *PQR* where *p*, *q* and *r* are the sides of the triangle, fill out the appropriate parts:

$$\sin P = \frac{\blacksquare}{r},\ \cos P = \frac{\blacksquare}{r}$$

$$\frac{\sin P}{\cos P} = \frac{\frac{\blacksquare}{r}}{\frac{\blacksquare}{r}} = \frac{\blacksquare}{\blacksquare} \qquad \textit{(Simplify.)}$$

$$\tan P = \frac{\blacksquare}{\blacksquare}$$

   Does your observation from 7 hold true in this case?

   Similarly, fill the parts in:

$$\frac{\cos P}{\sin P} = \frac{\frac{\blacksquare}{r}}{\frac{\blacksquare}{r}} = \frac{\blacksquare}{\blacksquare} \qquad \textit{(Simplify.)}$$

$$\cot P = \frac{\blacksquare}{\blacksquare}$$

   Does your observation from 7 hold true in this case, too?

Using the Library

1. There are so many famous mathematicians living today, it is difficult to single out only a few. The topics of interest to these people are generally far beyond the high school level. It is worthy to note however that the area of mathematics is still an expanding one, both from the aspect of pure mathematics and of the application of mathematics. The advancements in such fields as nuclear physics, computers, genetics, psychology, space exploration and medicine (to mention a few) have all been made possible through the continued advancements in mathematics.

   Put these events on your time line.

   Wiener Heisenberg, born – matrix mechanics
   John von Neumann, born – was considered the smartest man in the world
   Kitty Hawke
   Einstein – presented his Theory of Relativity
   Kurt Gödel, born – set forth a theorem challenging the very foundations of mathematics
   World War I begins
   Samuel Eilenburg, born – a topologist who claims he thinks best while lying down or riding on a subway
   League of Nations formed
   Stalin succeeds Lenin who died
   Stock Market crash
   Hitler came to power
   First atomic bomb dropped
   Nicholas Bourbaki
   World War II begins
   United Nations formed
   Sputnik I
   First man steps on the Moon

2. Research the pen name Nicholas Bourbaki. What was the purpose of this group?

3. Write a brief history of calculators and computers. What was "Babbage's Folly"? Why do people say computers think?

4. Write a defense for the statement "Without mathematics man would never have walked on the moon."

REVIEW EXERCISE 12

1. With the aid of a *sine-cosine tracer* find the values of:

   $\sin 30°$, $\cos 30°$, $\cos 60°$, $\sin 150°$, $\cos 240°$, $\cos 145°$, $\cos 320°$, $\sin 200°$, $\sin 350°$, $\sin 135°$.

   Check your findings using a set of tables.

2. For the sine curve, $y = \sin\theta$, what is happening to the trigonometric value y as the angle θ

   (a) is decreasing from 0° to 90°?
   (b) is increasing from 90° to 270°?
   (c) is at 360°?
   (d) is at −90°?

3. For the tangent graph $y = \tan\theta$, what is happening to the trigonometric value y as the angle θ

   (a) is increasing from −45° to 45°?
   (b) is at 90°?
   (c) is at 270°?
   (d) is increasing from −225° to −135°?

4. In what way do the graphs of

   (a) $y = \sin\theta$ and $y = \cos\theta$ differ?
   (b) $y = \sin\theta$ and $y = \sin 2\theta$ differ?
   (c) $y = \cos\theta$ and $y = 3\cos\theta$ differ?

5. What are the amplitude and period of each of the following trigonometric functions?

   (a) $y = \cos\theta$
   (b) $y = \tan\theta$
   (c) $y = 2\sin\theta$
   (d) $y = \tan\frac{1}{4}\theta$
   (e) $y = \cos 3\theta$
   (f) $y = \sin\frac{1}{4}\theta$
   (g) $y = 2\cos\frac{1}{3}\theta$
   (h) $y = \frac{1}{2}\sin 3\theta$.

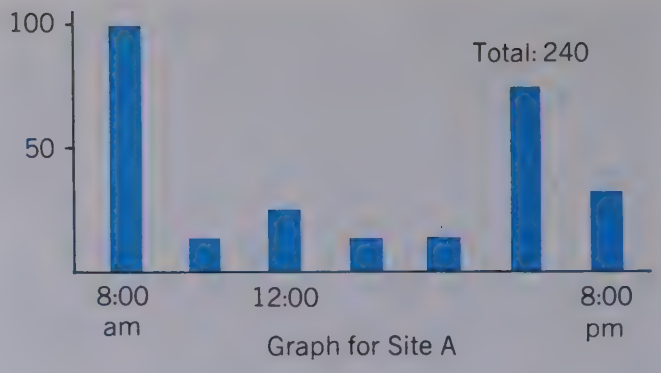
Your Job and Mathematics

Where to Locate Your Business

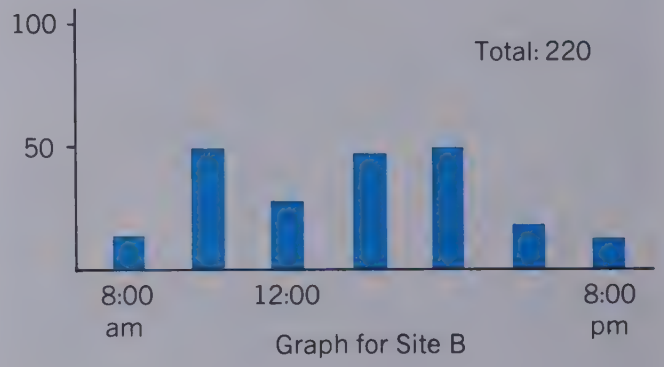
Let us assume you wish to start a business operation on a site in your community. Choose a business you would like to operate: a service station, a beauty parlor, a hairstylist shop, an art centre, a hamburger stand or an ice cream stand, etc. Choose three possible sites, then survey each area to see which would most likely be the best site. To do this you must record the number of potential customers that pass the site at different times of the day. See table below. Each survey must be under the same conditions. For instance, do not survey site A when it's raining and site B when it's sunny.

How will you identify potential customers? What other conditions need to be considered? Would you necessarily locate at the site with the highest number of potential customers? What factors would affect your decision?

If your graphs looked like Graph A and Graph B what location would you choose? Why?



Graph for Site A



Graph for Site B

For Service Station

	Site A	Site B	Site C
8:00 – 8:15			
10:00 – 10:15			
2:00 – 2:15			
4:00 – 4:15			
5:00 – 5:15			

For Hairstylist Shop

	Site A	Site B	Site C
9:30 – 9:45			
11:15 – 11:30			
1:30 – 1:45			
2:30 – 2:45			
3:45 – 4:00			
5:00 – 5:15			

Consider what times you will take the surveys. In the tables it is suggested counts be taken at 15-minute intervals. Discuss the need to consider the time for the survey. Why would the times be different for a service station and a hairstyling shop? Why would 15-minute intervals be sufficient? Is 15 minutes too long?

Chapter Thirteen

HOME OWNERSHIP

13·1 Introduction

Probably the most important financial decision you will make in your life is the decision to buy a house. You will commit a fairly large portion of your earnings to make this kind of purchase.

But not everyone decides to be a home owner. There are those people who decide to rent a house or an apartment.

How does one go about making a decision on home ownership? What are the reasons which determine whether one should buy a house or rent living accommodations?

Project

Project 1

Purpose

To set up guidelines which would assist one in making a decision on whether to buy a house or rent accommodations.

Method

1. Every member of the class, whether working singly or in small groups, gathers the required information. The information could be organized in the following manner:

OWNING A HOUSE	
Advantages	Disadvantages

RENTING A HOUSE OR APARTMENT	
Advantages	Disadvantages

2. Leave no stone unturned – search out information from all possible avenues (owners, renters, real estate agents, insurance agents, landlords, etc.). Visit these people or invite them to the school.

Questions

1. In a class discussion consider the value of each advantage and disadvantage offered by the students. Determine whether each will assist in the making of the decision to own or to rent.
2. Make a chart of your findings. Make a stencil of the chart for the purpose of duplicating copies of it for distribution to all members of the class.

EXERCISE 13·1

1. Can what is considered an advantage by one person be considered a disadvantage by another person (and vice versa)? Choose some points from the chart above and explain when this could be so.
2. What should be more important in making the decision (to own a house or to rent):
   (a) financial reasons or (b) personal reasons?
3. Do you agree with the following statement: "Only you can make the decision for yourself whether you should buy a house or rent"?

13·2 Costs of Purchasing a House

Projects

Project 1

Purpose

To become informed on all financial aspects relative to the purchase of a house.

Method

Resources

(i) Central Mortgage and Housing Corporation (CMHC) Government of Canada, Ottawa (or office in your town) for pamphlets on house buying.

(ii) Local Real Estate Board for all available literature concerning house buying.

(iii) Resource people and speakers.

1. Write to CMHC for information such as pamphlets and brochures on house buying.
2. Visit your local Real Estate Board for literature on house buying.
3. Invite a person (or persons) from a local Real Estate Board (Agency) or Mortgage Company to give a talk to your class. Choose a topic like "What Young People Should Know About House Buying" for your speaker. Give him a list of terms you wish to have explained:
   - Real Estate Agent
   - Down Payment
   - Mortgage (First, Second, other types)
   - Fees (legal, mortgage, etc.)
   - Mortgage companies
   - Lending Value (how determined?)
   - National Housing Act
   - Central Mortgage and Housing Corporation
   - Terms of mortgages
   - Title to property
   - Maximum percentage of earnings to monthly mortgage payments.
   - Depreciation and Appreciation
   - Who can and who cannot qualify for a mortgage?
4. Arrange for a discussion and question period after the talk.

Project 2

Purpose

To investigate the cost of buying a house.

Method

1. Have three groups of students from your class do this follow up.
2. Let group A choose to purchase a house valued under $20,000.
3. Let group B choose a house valued from $20,000 to $25,000.
4. Let group C choose a house valued over $25,000.
5. Each group should investigate what can be purchased with regard to:
   (a) area of living space,
   (b) size of lot and location in town,
   (c) extra features available,
   (d) exact cost of property,
   (e) down payment,
   (f) mortgage (term, rate of interest),
   (g) legal fees (what they cover).
6. Each group presents a written report on their investigations to purchase. They can be duplicated for distribution to each student in class. A verbal presentation of each report should be made. Discussions and questions following each report should be allowed.

13·3 Mortgages

Have committees of two or three students each visit various mortgage lending agencies to gather actual cost figures on specific mortgages. Then each committee prepares and presents a report of its findings. One of the committees should visit (or write to) Central Mortgage and Housing Corporation so that a comparison can be made with other lending institutions.

Let each committee choose a sale value of the house of approximately $20,000 (or some other agreed upon figure for your locality). In your study find out:

(a) what are the restrictions on obtaining a mortgage loan (earning power, age, financial standing such as bank accounts, debts, etc.);

(b) what is the minimum cash requirement for a mortgage loan;

(c) what is the rate of interest on the loan;

(d) what is the term of the loan;

(e) what are the monthly payments for repayment of the loan; (Are they uniform payments?)

(f) whether the loan can be repaid in a lump sum at a later date;

(g) if property taxes must be included in the monthly payments;

(h) if fire (and other) insurance is a requirement to obtain a loan.

13·4 Amortization of a Mortgage

The word *amortization* means "the repayment of a loan by a sequence of periodic payments". Repaying a mortgage, which is a loan on property, is really paying the loan company an annuity. The mortgage is the present value of such an annuity.

There are two common ways of amortizing a mortgage. One is on a decreasing payment plan, and the other by uniform payments.

On a decreasing payment plan a fixed amount (to decrease the principal) is paid periodically plus the interest on the unpaid balance for that period. In this way the first payment is the largest because the interest calculated is on the whole loan (mortgage) value. As the principal decreases with each payment so does the subsequent interest for each period.

Amortizing by uniform payments is the repayment of the loan (mortgage) in fixed amounts at the end of each period of time. The fixed amount includes the payment on the principal and an interest payment. At the beginning of term of the loan the interest portion of the fixed amount is very high in comparison to the payment on the principal. As payments progress and the principal is reduced, the interest payments decrease from period to period while the payments on the principal increase.

Example 1

Mrs. and Mr. Houseowner gave a loan company a second mortgage for $2000 at 12% per annum. The mortgage was to be paid off on a decreasing payment plan of $250 every three months plus interest. The agreement was signed on April 1. Draw up an amortization table for the payment of this mortgage. Find the total interest to be paid.

Solution

DATE	INTEREST	PAYMENT ON PRINCIPAL	TOTAL OF PAYMENT	BALANCE TO PAY
Apr 1				$2000.00
Jul 1	$60.00	$250.00	$310.00	1750.00
Oct 1	52.50	250.00	302.50	1500.00
Jan 1	45.00	250.00	295.00	1250.00
Apr 1	37.50	250.00	287.50	1000.00
Jul 1	30.00	250.00	280.00	750.00
Oct 1	22.50	250.00	272.50	500.00
Jan 1	15.00	250.00	265.00	250.00
Apr 1	7.50	250.00	257.50	0.00
TOTAL	270.00			

Total interest to be paid is $270.00.

To do Examples 2 and 3 is beyond the mathematics done in this book. Some very detailed and far extending tables are required. Books of tables including Monthly Amortized Mortgage Payments are available for such purposes. A very small section is excerpted from such tables to serve our needs here.

Monthly Payment of Principal and Interest per $1000 of Loan Value

TERM IN YEARS	RATE OF INTEREST PER ANNUM				
	8%	$8\frac{1}{4}$%	$8\frac{1}{2}$%	$8\frac{3}{4}$%	9%
3	31.2764	31.3879	31.4996	31.6114	31.7233
5	20.2142	20.3299	20.4459	20.5622	20.6787
7	15.5214	15.6419	15.7628	15.8841	16.0058
10	12.0641	12.1918	12.3202	12.4492	12.5789
12	10.7533	10.8859	11.0192	11.1532	11.2880
15	9.4815	9.6211	9.7616	9.9029	10.0452
20	8.2836	8.4340	8.5856	8.7382	8.8919
25	7.6321	7.7923	7.9536	8.1161	8.2798

Example 2

Mr. Landlord obtained a loan from a mortgage company to buy an apartment block. The value of the loan was $40,000. The term for that loan was 12 years and the rate of interest was $8\frac{1}{4}$% per annum.

(a) Find the uniform monthly payment of principal and interest Mr. Landlord has to make.

(b) What will the total of the payments be?

(c) How much is the total interest on this loan?

Solution

(a) On a $1000 loan value at $8\frac{1}{4}$% per annum for 12 years the monthly payment of principal and interest is $10.8859. (See preceding table.)

On $40,000, the monthly payment of principal and interest is

$$\frac{40{,}000}{1000} \times \$10.8859 = \$435.44$$

(b) In 12 years the total number of monthly payments made will be $12 \times 12 = 144$.
Then, total of the payments will be

$$144 \times \$435.44 = \$62{,}703.36.$$

(c) Total interest on the loan is

$$\$62{,}703.36 - \$40{,}000 = \$22{,}703.36$$

Example 3

Mary and John bought a house valued at $29,500. They made a down payment of $8,000 and gave a mortgage on the balance for 25 years at $8\frac{3}{4}\%$ per annum.

(a) What was their monthly payment?

(b) What will the total cost of the house be?

(c) How much will they pay in interest charges?

Solution

(a) From the table, the monthly payment on $1000 is $8.1161.

The mortgage value is

$$\$29{,}500 - \$8{,}000 = \$21{,}500.$$

The monthly payment on the mortgage is

$$\frac{21{,}500}{1000} \times \$8.1161 = \$174.50.$$

(b) Number of payments in 25 years is

$$25 \times 12 = 300.$$

Total of payments in 25 years is

$$\$174.50 \times 300 = \$52{,}350.$$

Total cost of the house will be

$$\$52{,}350 + \$8{,}000 = \$60{,}350.$$

(c) Interest charges on the mortgages are

$$\$60{,}350 - \$29{,}500 = \$30{,}850.$$

EXERCISE 13·4

1. Mr. and Mrs. Owens required a loan for renovations. They gave a second mortgage for $1800 to be paid off on a decreasing payment plan of $200 every 4 months plus interest at 12% per annum. The first payment was due January 1. Draw up an amortization table for the repayment of this loan.

2. Alice and Bob gave a second mortgage for $1500 at 12% per annum. The mortgage is to be paid off by making monthly payments of $75 each plus interest. Set up an amortization schedule for the payment of this mortgage. First payment was due August 1. How much interest will have been paid when the mortgage is retired?

3. For each of the following mortgages find the uniform monthly payment of principal and interest. (Use tables in this section.)

	Mortgage	*Term*	*Rate of Interest*
(a)	$ 6,000	3	9%
(b)	$10,000	7	$8\frac{3}{4}\%$
(c)	$12,500	15	8%
(d)	$18,600	10	$8\frac{1}{2}\%$

4. Lucy and Jack bought a house for $33,000. They made a down payment of $7,000 and gave a mortgage on the balance at $8\frac{1}{4}\%$ for 25 years. (Use tables in this section.)
   (a) How much was their monthly payment?
   (b) How much was the total cost of the house?
   (c) What was the total of the interest charges?

5. Yvonne and Frederick bought a house for $18,750. They made a down payment of $4500 and gave a mortgage on the balance at $8\frac{1}{2}\%$ for 20 years. (Use table in this section.)
   (a) How much was their monthly payment?
   (b) How much was the total cost of the house?
   (c) What was the total of the interest charges?

6. What is a *second mortgage*?
   Why should interest rates on second mortgages be higher than on first mortgages? Discuss.

   Collect from newspapers and advertisements, information on second mortgages. Is it easy to get a second mortgage?

   If you own a house with a first mortgage can you borrow money by obtaining a second mortgage? Discuss.

7. What are the (a) advantages and (b) disadvantages of buying a new house over a secondhand house?

13·5 Maintenance Costs

Property Taxes

Have a committee of two visit City Hall or municipal office for the purpose of obtaining information on property taxation. If possibile, invite someone from the Taxation Department to visit your class and present a talk on "How Property Taxes are Levied". Give the speaker the following topics to cover in his talk.

- Assessment of property (Who assesses?
- What determines the assessed values?
- What are some examples of assessments?)
- Are assessed values related to real estate sale values?
- Mill rate (How set? How calculated? Examples of rates in your locality.)
- Taxation notices
- Examples of property taxes
- How improvements to property affect taxes

Refer to **Math Probe 3**, Chapter 10.

Property Insurance

Invite a property insurance agent to present a talk to your class on all aspects of property insurance. Ask him to cover such parts as:

(a) dwelling damage (by fire, explosion, wind, hail, etc.)

(b) household property damage (by smoke, water, freezing, etc.)

(c) personal liability (bodily injury and property damage)

(d) theft and burglary

(e) replacement cost

(f) examples of premiums for specific coverage

(g) others.

Refer to **Book 2**, Section 4·12.

Utilities (Electricity, Water) and Heating

From bills received at home collect costs for utilities and heating. Find the average costs for each utility and fuel for heating in your locality for one month. In a class, discuss:

- Are some utility and heating costs excessively high?
- What factors contribute to high costs?
- What remedies can be taken to lower the costs of utilities and heating?
- What factors cannot be altered to save on costs?

Other Costs

Consider other maintenance costs in home ownership. In a class discussion enumerate these various expenses. Discover whether these costs vary from dwelling to dwelling. Are there precautions to be aware of and steps to take in order to prevent these maintenance costs from becoming excessive?

If any parts of the discussion cannot be resolved in class to the satisfaction of the class members, committees of two may visit builders, lumber yards, paint stores, etc. for up-to-date and reliable information.

13·6 Calculation of Fair Rental Value

A landlord has to take into account all the expenses he is expected to pay for annually in calculating the rent he must charge his tenants. Then he is justified in adding to the expenses a return on his investment.

What are the expenses that a landlord takes into account? First, there are the property taxes and insurance. Then, come the repair and depreciation costs. Of course, upkeep expenses like redecorating and painting add to the total cost. If he is renting a number of suites in an apartment block he will then have to collect for the utility charges (water, gas, hydro).

Example 1

A landlord rents a house valued at $24,000. His taxes are $580 a year. He pays $108 in property insurance for 3 years coverage. He estimates his repair and depreciation costs at 2% of the value of the house. He redecorates and paints the house every 3 years at a cost of $630. He expects to receive a return of 6% on the value of his investment. Calculate a fair rental value for this property.

Solution

The yearly costs to the landlord are:

Taxes	$580
Property insurance ($\frac{1}{3}$ of $108)	36
Repair & depreciation ($\frac{2}{100} \times 24{,}000$)	480
Redecorating & painting ($\frac{1}{3} \times$ $630)	210
Return on investment ($\frac{6}{100} \times 24{,}000$)	1440
Total of yearly costs	$2746

A fair monthly rent for this property is about

$$\$2746 \div 12 \simeq \$230.$$

Example 2

Mr. Lewis owns an apartment block with suites for rent. He estimates a two-bedroom suite is valued at $13,500. He pro rates the property taxes for the suite at $280 a year. He estimates his repair, depreciation and upkeep costs for the suite at $430 a year. The property insurance premium is about $90 for 3 years. The total utility costs are approximately $300 annually. He expects a return of 7% on his investment in the property. Calculate a fair rental value for this suite.

Solution

The annual costs to Mr. Lewis are:

Taxes	$280
Repair, depreciation & upkeep	430
Property insurance ($\frac{1}{3}$ of $90)	30
Utility costs	300
Return on investment ($\frac{7}{100} \times$ $13,500)	945
Total annual costs	$1985

A fair rental charge for this suite is about

$$\$1985 \div 12 \simeq \$165.$$

EXERCISE 13·6

1. Mr. Smith invested $22,500 in a house which he is offering for rent. Taxes on this property are $540 annually. He estimates his repair, depreciation and upkeep costs will be about 3% of the value of the house every year. His insurance premium is $96 every three years. He expects a return of $5\frac{1}{2}$% on his investment. What is an approximate fair rent Mr. Smith can charge?

2. James Dodd inherited a house. Since he had no need to live in it he offered it for rent. He had an appraisal value of $18,000 set on the house. The taxes on the house were $495. He was counselled to charge $1\frac{1}{2}$% of the appraisal value for depreciation costs. He estimated his repair and upkeep costs at $400 yearly. He expects a return of 6% on the value of the house (his asset now). What should be an approximate fair rental charge on this house?

3. Lucille Brown invested $46,000 in an apartment block which comprised four identical suites. She discovered that the annual property taxes were $1260. She was advised that her depreciation costs would amount to approximately 2% of the value of the property. From the past owner she learned that repair and upkeep costs averaged $1100 annually. The 3-year insurance premium was $290. Records from the past owner indicated that annual utility costs were approximately $1240. If she expects a return of 7% on her investment, what will a fair rent charge per suite be?

4. Can you think of any other expenses that might be included in establishing fair rental value? Discuss.

REVIEW EXERCISE 13

1. State briefly whether you would prefer to purchase your home (assume you have the down payment) or rent. Use the information you have gained in studying this chapter to justify your decision.

2. Ron and Sandy required $3000 for renovations and improvements to their property. They gave a loan company a second mortgage for the $3000 at 15% per annum. They planned to pay off the debt at $300 every three months plus interest. The first payment was due October 1. Draw up an amortization schedule for the payment of this mortgage. What is the total interest for this amortization?

3. Jane and Bob bought a house valued at $24,500. They made a down payment of $4000, and gave a mortgage on the balance for 25 years at $8\frac{1}{2}\%$ per annum. (Use table in Section 13·4.)

   (a) What was their monthly payment on the mortgage?

   (b) What will the total cost of the house be?

   (c) How much will they pay in interest charges?

4. Sally and Jack invested $40,000 in a duplex. The two suites in the duplex were of identical design and construction. They planned to live in one suite and rent the other. The property taxes were $680. The insurance premium was $135 for a three year period. They estimated the repair, depreciation and upkeep costs for the duplex would be $900 a year. The utility costs for the year approximated $530. They expected a return of 7% on the value of the suite to be rented. Calculate what a fair monthly rental value for the suite should be.

   Note! Taxes, insurance premium, utility costs, depreciation (etc.) costs are given for both suites. Is it fair to use each total figure in determining the monthly rent?

Chapter Fourteen

AND FOREIGN EXCHANGE EXCISE TAXES, DUTIES

14·1 Everybody Pays Taxes

The Canadian government taxes most products manufactured in Canada and most products imported. A few products exempted are foods, farm and fishery equipment and machinery, educational and religious equipment and goods. There are a number of different taxes: federal sales tax, excise tax, excise duty, custom duties (tariffs) and provincial sales tax.

Federal Sales Tax

The Federal Government levies a sales tax on a wide variety of items. The rate levied varies as to the type of item sold. The tax is calculated on the manufacturer's selling price. If the item has been imported then the tax is based on the duty-paid value. The duty-paid value includes the cost of the goods *plus* the customs duties charged against the items.

Most often the manufacturer collects the federal sales tax for the government. This tax is often termed a *hidden* tax since the consumer does not see the tax applied. Sometimes the wholesaler collects the federal tax and in turn pays it to the Canadian government. If the product is imported custom officials collect the sales tax at the time of entry.

The federal sales tax varies, for most products it is 12%, but on building products and equipment it is 11%.

An excise tax is imposed by the Federal Government on a few special luxury items. Watches, jewellery, wine, cigarettes are a few so taxed items. In 1971 the tax on radios and televisions was removed. The taxes are collected by the manufacturer (if Canadian made) and by customs officials if imported. See Table 1.

Table 1

Excise Tax	
Product	Rate
Cigarettes, for each 5 or fraction thereof	.03¢
Cigars	15%
Clocks and watches, jewellery	10%
Compacts, vanity cases and cosmetics	10%
Record Players	15%

An excise duty is charged on spirits and tobacco manufactured in Canada. See Table 2.

Table 2

Excise Duty	
Product	Rate
Cigarettes, for each 5 or fraction thereof	$ .02
Cigars, per thousand	$ 2.00
Tobacco, raw leaf, per pound	$ .10
Beer, per gallon	$ .38
Brandy, per gallon	$12.25

On imported goods listed in Table 2 an additional tax is levied which is approximately equivalent to this excise duty. Hence the Canadian manufacturers are not at a tax disadvantage.

Example 1

Calculate the total cost to the wholesaler of these Canadian products priced by the manufacturers as shown. Sales tax is 12% on all items.

(a) 30 bicycles at $56.50
(b) 5000 cigarettes at 5¢
(c) 100 rings at $30.75

Solution

Find the total manufacturers price of each item and calculate the taxes when applicable in this order:

(i) Excise duty, (ii) Excise tax, (iii) Sales tax on duty-paid value.

(a)	30 bicycles at $56.50	= $1695.00	
	Federal sales tax of 12%	= 203.40	
	Total cost		$1898.40
(b)	5000 cigarettes at 5¢	= $ 250.00	
	Excise duty $\frac{5000}{5} \times \$.02$	= 20.00	
	Duty-paid value	$ 270.00	
	Excise tax $\frac{5000}{5} \times \$.03$	= 30.00	
	Sales tax, 12% of $270	= 32.40	
	Total cost		$332.40
(c)	100 rings at $30.75	= $3075.00	
	Excise tax 10% of $3075	= 307.50	
	Federal sales tax 12% of $3075	= 369.00	
	Total cost		$3751.50
	Total cost of all items		$5982.30

Some Federal Responsibilities

The federal government needs money in order to meet its financial responsibilities. Discuss each responsibility illustrated and explain how each is a revenue or an expenditure or both for the government.

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Provincial Sales Tax

Each province is empowered to levy a sales tax which is collected at the retail level. In 1971 the sales tax in Ontario was 5%, in Quebec 8% and in Newfoundland 7%. The provincial sales tax is based on the retail selling price of the goods *after* the federal tax has been applied. As with the federal tax, certain items are exempt from the provincial sales tax. Generally, food (except in a restaurant), children's clothing and newspapers are exempt.

To assist the clerk in a retail store a table showing the amount of tax for the various selling prices is posted close to the cash register. A table for a 5% sales tax is shown.

Amount of Sales			*Tax*
$		$	¢
.00	to	.09	0
.10	to	.29	1
.30	to	.49	2
.50	to	.69	3
.70	to	.89	4
⋮	⋮	⋮	⋮
4.90	to	4.99	25

Sales tax is calculated to the nearest cent. One half cent is counted as a cent. The sales tax on a sale greater than that shown on the table is calculated as follows.

(1) Sales tax = rate of tax × amount of sales

The amount payable is found by

(2) Amount payable = amount of sales + sales tax

Example 1

Find the amount payable if the amount of the sales is $15.45 and the sales tax is 5%.

Solution

Sales tax = rate of tax × amount of sales
= .05 × 15.45 = .7725
Sales tax is 77¢
Amount payable = 15.45 + .77
= $16.22

Example 2

Calculate the amount a consumer pays for an article if the manufacturer produces it for the market at a cost price of $38.50. The federal sales tax is 12% and the provincial sales tax is 8%.

Solution 1

First calculate the federal sales tax.

Sales tax = 38.50 × .12
= 4.62
Price to retailer = 38.50 + 4.62
= 43.12

Next calculate the provincial sales tax.

Sales tax = 43.12 × .08 (Use price to
= 3.45 retailer)
Price to consumer = 43.12 + 3.45
= 46.57

The consumer pays $46.57.

Solution 2

This can be done in two steps.

1. The price to the retailer is 112% of the manufacturer's cost price, therefore
   Price to retailer = 38.50 × 1.12
   = 43.12
2. The price to the consumer is 108% of the retailer's price, therefore
   Consumer's price = 43.12 × 1.08
   = $46.57

You may use whichever method suits you.

EXERCISE 14·1

1. Calculate the excise duty and tax and the sales tax on these items: (sales tax 12%).
   (a) 1200 cigars at 12¢ a cigar
   (b) 100 watches at $32.60 each
   (c) 50 gallons of Brandy at $15.00
   (d) 3000 cigarettes at 2¢.

2. Complete the 5% sales tax table to sales of $5.00.

3. Find the sales tax for your province. Set up a table similar to that in question 1 for the sales tax on the first five dollars or parts thereof.

4. Why does the federal government levy sales tax? Why do some provincial governments levy sales tax while at least one does not?

5. Find the sales tax on a car priced without tax by the manufacturer at $2140 if the federal levy is 12%.

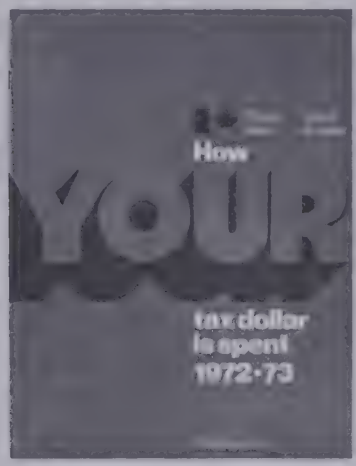
6. Find the sales tax on a pair of skis priced to the retailer at $72 if the provincial sales tax is 8%.

7. Find the missing information in each case.

	Price without tax	Federal tax levy	Price to retailer	Provincial tax levy	Cost to consumer
(a)	$ 35.50	9%	?	none	?
(b)	$ 74.35	none	?	6%	?
(c)	$ 164.52	12%	?	8%	?
(d)	$ 356.78	15%	?	5%	?
(e)	$3,540.50	11%	?	7%	?
(f)	?	12%	84	6%	?
(g)	?	12%	89	5%	?
(h)	?	12%	?	5%	$94.50
(i)	?	12%	?	5%	$61.22

8. A wholesaler imports a transistor radio from Japan. The cost of the goods to the wholesaler is $24 plus a customs charge of 25% of cost. The federal sales tax is 15%. The provincial sales tax is 6%. What is the price to the consumer?

9. Send for your free copy of this book. Write to Information Canada, 171 Slater Street, Ottawa 4, Ontario.



Identify the four services provided by the Federal Government that take the largest portion of your tax dollar.

What percent of your tax dollar does
(a) education assistance take
(b) recreational and cultural programs take?

How many cents out of each dollar go back to the provinces in cash payments?

14·2 Custom Duties

Custom duties are essentially a protective tax. Normally they are imposed by a country in order to protect the industries within that country from the competition of industries in foreign countries. The Canadian government publishes a tariff book in which is listed the tariff on every possible classification of goods. There are three classes of tariffs according to the countries from which the goods come.

The British Preferential Tariff

This tariff structure is imposed on all goods produced or manufactured in and shipped directly from Britain, British Colonies or Protectorates or territories under British Trusteeship. It is the lowest of the three classes. It is important to note that the items must be conveyed without transshipment to a country included by this tariff agreement.

The Most Favoured Nation Tariff

This tariff structure applies to those countries with which Canada has this trade agreement. Countries included by this agreement are Germany (Western), Belgium, Brazil, Cuba, France, Italy, Japan, Mexico, Netherlands, Russia, Sweden, Spain, U.S.S.R., U.S.A. Again items must be manufactured or produced in one of the countries covered by this agreement and must not be transshipped via a country not covered by this or the British Preferential Tariff Agreement.

The General Tariff

This rate covers those countries not listed in the first two agreements. Hungary, Romania and East Germany are in this group.

Special trade agreements are entered into by Canada and certain countries from time to time for specific items and with specific reasons. These will not be discussed here. You may wish to research this area.

Import tariffs are imposed by the Canadian government on three different bases.

Ad Valorem Duty

This duty is assessed as a percentage of the fair market value in the country of export. Several factors go into calculating the *fair market value*. It is often difficult to ascertain this value. Two of the factors are the actual selling price and the actual market price. If the selling price is substantially lower than the market value then a *dumping duty* is imposed.

Example 1

What is the price in Canada of a Rolls Royce valued at $25,600 in Britain, if the duty is 12% of fair market value? Shipping charges are $346.

Solution

Selling price	$25,600
12% duty	3,072
Shipping	346
Price in Canada	$29,018

Dumping Duty

Canada, in 1904, was the first country to introduce genuine anti-dumping tariff legislation. It stated that goods that are exported to Canada at prices below fair market value in the exporting country are to be assessed in two stages:

(i) for duty purposes at the fair market value in the exporting country, and

(ii) for an additional duty equal to the difference between the invoice and fair market value price.

Dumping duty is designed primarily to prevent a country from dumping goods overstocked in that exporting country, onto the Canadian market.

Example 2

Japan, overstocked in clocks, exports some to Canada. The invoice quotes each at $75.00. The fair market value in Japan is $112.50. Canada imposes a normal 30% ad valorem duty and a dumping tax according to the above statement. Calculate the gross price of each clock in Canada.

Solution

1. Calculate the ad valorem tax:

   Tax = fair market value × 30%
   = 112.50 × .30 = 33.75

2. Find the duty-paid value:

   Duty-paid value is made up of
   (fair market value + tariff tax)
   Duty-paid value = $112.50 + $33.75 = $146.25

3. Calculate the excise tax:

   Excise tax is calculated on duty-paid value:
   Excise tax = duty-paid value × rate of tax
   = 146.25 × 10% = $14.63

4. Calculate the dumping tax:

   Dumping tax = fair market value − invoice price
   Dumping tax = $112.50 − $75.00 = $37.50

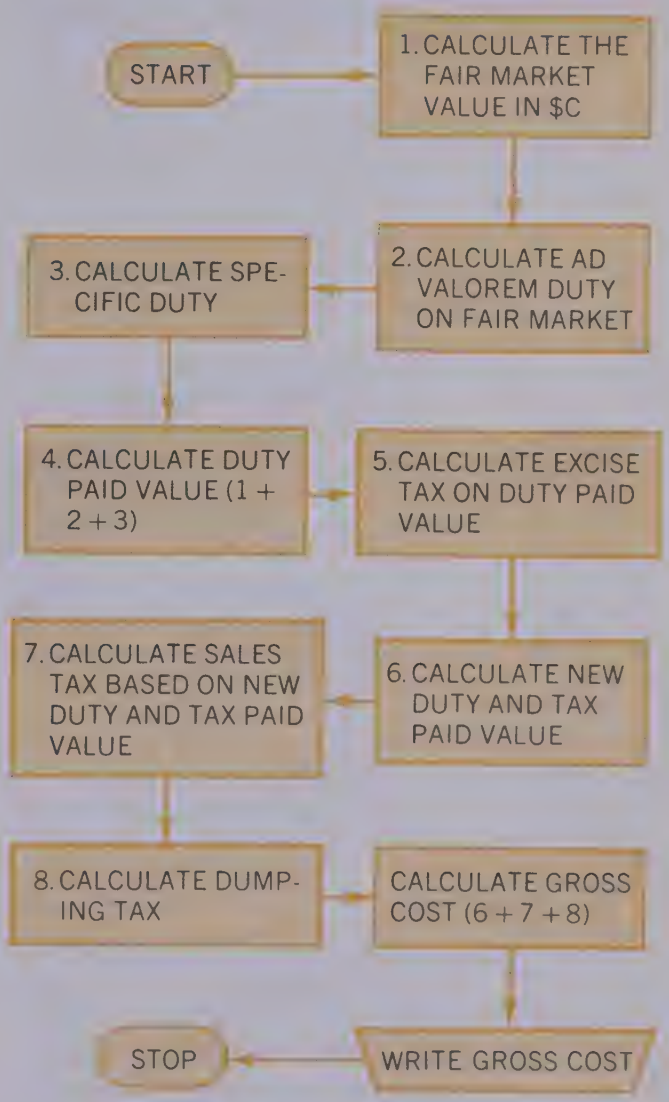
5. Gross price = invoice price + ad valorem tax
   + excise tax + dumping tax
   = $75.00 + $33.75 + $14.63 + $37.50
   = $160.88
   The gross price is $160.88.

NOTE! We did not indicate a sales tax on the clocks.

Specific Duty

This is a direct tax levied on the quantity of the item. It is a fixed charge per pound (or kilogram), per foot (or other lineal measure), or per unit such as per dozen, per gross, per 100, etc., or per gallon (or other volume units). Some items carry both ad valorem and specific duty. It is even possible that an item could be charged ad valorem, specific and dumping tariff.

This flow chart gives the order for calculating taxes, duties, and tariffs if and when each is individually applicable for the specific item being imported.


START
1. CALCULATE THE FAIR MARKET VALUE IN $C
2. CALCULATE AD VALOREM DUTY ON FAIR MARKET
3. CALCULATE SPECIFIC DUTY
4. CALCULATE DUTY PAID VALUE (1 + 2 + 3)
5. CALCULATE EXCISE TAX ON DUTY PAID VALUE
6. CALCULATE NEW DUTY AND TAX PAID VALUE
7. CALCULATE SALES TAX BASED ON NEW DUTY AND TAX PAID VALUE
8. CALCULATE DUMPING TAX
CALCULATE GROSS COST (6 + 7 + 8)
WRITE GROSS COST
STOP

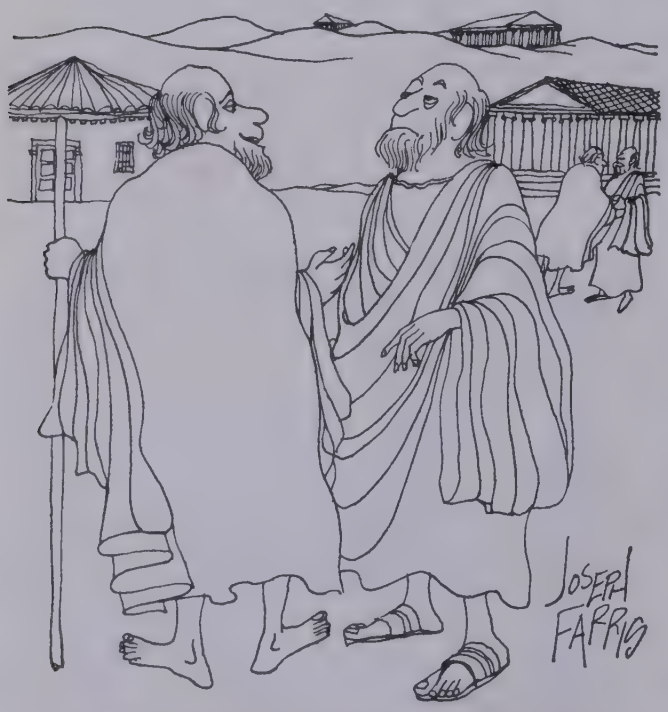
EXERCISE 14·2

1. Use the library or dictionary to find the origin or meaning of the term *ad valorem* as applied to this chapter.

2. Use your library to determine what special trade agreements Canada currently has. Determine if you can why these special agreements were entered into by the respective countries.

3. Why do you think Canada has different trade agreements with various countries? Discuss this in groups of 8 or 10 students. Is there common agreement? Summarize your group's conclusions. Report these conclusions to the rest of the class in some suitable way.

4. The B.T.B. Construction Company imported a special crane for high-rise construction. The crane has a fair market value of $3945. The ad valorem duty rate is 8%.
   (a) Calculate the duty paid value.
   (b) There is a sales tax of 11%. Calculate the final price of the crane without shipping charges.



"It's simple. The way to curb inflation is to float the drachma."

5. Following is a list of items, the exporting country, the quantity and price per unit. Calculate the tariff on each item. (Do not consider excise tax or duty.) Use Table 3.

ITEM	QUANTITY	PER UNIT FAIR MARKET VALUE	COUNTRY
Car	10	$2850	Germany
Car	10	$2850	Britain
Carpet, oriental	300 sq ft	$8.50 per sq ft	Spain
Carpet, oriental	300 sq ft	$8.50 per sq ft	Britain
Carpet, oriental	300 sq ft	$8.50 per sq ft	Romania
Wine	144 qt bottles	$6.50 / qt	Britain
Wine	144 qt bottles	$6.50 / qt	Hungary
Wool blankets	500 lb	$6.40 / lb	Britain
Wool blankets	500 lb	$6.40 / lb	Sweden

Table 3

Custom Tariff Schedule

ITEM	BRITISH PREFERENTIAL	MOST FAVOURED NATION	GENERAL
Automobiles	free	$12\frac{1}{2}\%$	$27\frac{1}{2}\%$
Bicycles	20%	25%	30%
Cameras (of type made in Canada)	$7\frac{1}{2}\%$	22%	30%
Carpets, orientals	25%	25%	40%
and per sq ft	–	5¢	20¢
Cigars	10%	10%	25%
and per lb	$1.50	$1.50	$3.50
and per 1000	$1.00	$1.00	$1.00
Clocks	15%	30%	35%
Guitars	15%	$17\frac{1}{2}\%$	30%
Phonograph records	free	5%	25%
Sterling silverware	$17\frac{1}{2}\%$	$27\frac{1}{2}\%$	45%
Wine, champagne	free	free	30%
per doz qt bottles	$9.30	$4.00	$9.30
Wire, barbed	free	free	free
Wool blankets	20%	25%	40%
and per pound	5¢	15¢	30¢
Tires, pneumatic	free	15%	45%

6. Two cars are imported, the Jaguar XJ64.2 automatic has an invoice price in England of $7340 (Canadian) while the BMW 2800 automatic has an invoice price in Germany of $7010 (Canadian). The Jaguar is admitted to Canada with a sales tax of 12% while the BMW has a duty of 15% and a sales tax of 12%.
   (a) Explain why one has duty and the other does not.
   (b) Which car cost the most in Canada? What is the difference in the two prices in Canada?

7. A carpet dealer imports carpet from England, Mexico and Romania. Use the tariff schedule.

Country	Invoice price	Sq feet
England	$3250	1600
Mexico	$2670	3218*
Romania	$1790	830

*Dumping duty applied. Fair market value = $4288

There is also a sales tax (federal) of 12%.

   (a) What is the ad valorem tax on each order?
   (b) What is the specific tax on each order?
   (c) What is the dumping duty?
   (d) Calculate the duty-paid value on each order.
   (e) What is the sales tax on each order?
   (f) Calculate the total cost on each order.

8. Wool blankets are imported from Italy. The invoice price on 3456 pounds is $7632 (Canadian). Calculate the gross cost of the blankets if there is a 10% sales tax (federal).

9. Ten thousand cigars at 12¢ each, weighing 300 lb., are imported from the U.S.A. Calculate the total price paid in Canada after tariff, excise tax, excise duty, and sales tax of 12% are calculated. Use tables given in this chapter.

10. The Pacific Importers Company agrees to buy from North Borneo 100 native masks for $4200. The customs charge is 35%. The federal sales tax is 12%. The importer's markup on costs is 30%. The importer consigns part of the shipment to a retailer who, in turn, has a markup on costs of 35%. The provincial sales tax is 6%. What is the price of one native mask to a consumer?

A large part of import-export shipping is now done in these containers.

11. Refer to question 10. What percent of the price the consumer paid is tax? is markup? What percent of the price the consumer paid did the producers of the mask get?

12. Refer to question 10. Do you think there may have been more *middlemen* than those listed in the question? *Middlemen* is the term often applied to all the businesses between the producer and the consumer.

13. Refer to question 10. What would the price of a mask to the consumer be if both the wholesaler and the retailer had a margin of 30% of sales?

14. Draw a rectangular graph, or a circular graph, to represent the proportions of the selling price that make up the various amounts that go into the final selling price. Refer to question 11.

15. A cost of a new toy to a retailer is $20. He has a markup of 30%. Then he finds the toy does not sell so he marks down the toy 30%. What is the sale price of the toy?

16. Is the sale price of the toy in question 15 $20? If not, can you account for the difference since there was a markup of 30% and a markdown of 30%?

17. In question 15, if the dealer had used a margin of 30% on sales, then effected a markdown of 30%, would the sale price be $20? Explain.

14·3 Foreign Exchange

In the previous section we discussed the value of items in foreign countries. For instance we cited the case (Example 1) of Japan exporting clocks to Canada. The example quoted the Japanese fair market value in Canadian dollars when actually the fair market value would be quoted in Japanese *yen*, the unit of currency in Japan. The Japanese quotation would have to be converted to the Canadian dollar equivalent.

One yen in 1972 was equivalent to $0.0032 Canadian or $1 Canadian was equivalent to 313 yen (correct to nearest unit). (Sometimes Canadian dollar is written $C.)

This section will deal with foreign currency and instances wherein converting from one currency to another would be employed.

Project

Project 1

Purpose

To become acquainted with units of foreign currency.

Method

Arrange a survey of the class to record

(a) the number of students who have visited outside of Canada,

(b) the countries visited by members of the class.

To the teacher: If insufficient data is obtained survey the students for the countries they would like to visit. Ask each to explain why they would want to visit the country indicated. Then proceed with part (c).

(c) Name the unit of currency of each country listed.

(d) Name the various coins or fractions of units used in each country. Use the library if necessary. For example, in Britain the pound is the unit and the pound and penny are the two units used.

When a Canadian visits a foreign country he must use the currency of that country. When a firm in Canada purchases an article from another country the article must be paid for in the currency of that country. In this day, millions of people are travelling from country to country and international trade is essential to the well-being of each country's economy. Hence it is necessary that a person or firm be able to convert money from one currency to another. These foreign exchange transactions are usually made through the foreign exchange departments of chartered banks.

Banks maintain current Foreign Exchange Rates. In simple terms, the rates usually show a bid (buying price the bank will pay) and an ask (price at which the bank will sell). The difference between the two prices reflect the bank's commission.

Foreign Exchange Rates, Dec. 10, 1971

Bid	Country	Currency	Ask
$99\frac{3}{32}$	United States	Dollar	99.90
2.51	United Kingdom	Pound	$2.56\frac{1}{2}$
$1.21\frac{1}{2}$	Australia	Dollar	$1.22\frac{3}{4}$
.0425	Austria	Schilling	.0445
.02210	Belgium	Franc	.02225
.1948	Brazil	Cruzeiro	.1962
.1445	Czechoslovakia	Crown	.1455
.1920	France	Franc	.1930
.1430	Denmark	Kroner	.1437
.3060	Germany	Mark	.3069
.003100	Japan	Yen	.003250
.0811	Mexico	Peso	.0818
.2572	Switzerland	Franc	.2578
.3069	Netherland	Guilder	.3080

Example 1

Mr. and Mrs. Hans Christensen, in preparation for a trip to Denmark, converted $300 Canadian to Kroners. How many Kroners did the bank sell them?

Solution

Since the bank is selling the Kroners we use the *Ask* rate.

$$\$.1437 \text{ Canadian} = 1 \text{ Kroner}$$
$$\$300 \text{ Canadian} = x \text{ Kroners}$$
$$x = 2087.68 \text{ Kroners}$$

Example 2

Mr. and Mrs. Bosta returned from Switzerland. They went to a bank to convert 250 Swiss francs to Canadian dollars. How many did they receive?

Solution

Since the bank is buying Swiss francs the rate used is the *Bid* price.

$$1 \text{ Swiss franc} = \$.2572 \text{ Canadian}$$
$$250 \text{ Swiss francs} = \$x \text{ Canadian}$$
$$x = \$64.30$$

Mr. and Mrs. Bosta received $64.30.

Many newspapers carry foreign exchange rates showing only the value of each currency on the market. This chart shows the value on the Foreign Exchange Market (which operates similar to a stock exchange) of an Austrian schilling. Since most countries have currency systems based on a decimal system, it is fairly safe to calculate to three decimal places and round to two places when calculating exchanges.

EXCHANGE

MONTREAL --
Foreign exchange rates:
Australia dollar 1.22
Austria schilling .0435
Belgium franc .02300
Brazil cruzerio .1960
Bulgaria lev .9614
Czechoslovakia crown .1450
Denmark kroner .1430
France franc .1930
Germany mark .3075
Hungary forint .0867
Italy lira .001700
Mexico peso .0815
Netherlands guilder .3080
Norway kroner .1505
Poland zloty .2519
Romania peseta .0155
Sweden kroner .2065
Switzerland franc .2580
United States dollar 100½
United Kingdom pound 2.57
Venezuela bolivar-free .2418
Yugoslavia dinar .0604

Quotations in Canadian funds

EXERCISE 14·3

1. Obtain from a current newspaper the Foreign Exchange Rate similar to that shown earlier. Obtain from a bank the current bid and ask quotations. Compare the two sets of rates. What do you observe? Can you account for the difference? You may wish to ask the bank manager or accountant.

Use the bid and ask quotations in this section for the remaining questions.

To the teacher: If the current rates are appreciably different from the rates given in this book it may be advisable to get the current foreign exchange rates from a bank.

2. Jimmy and Suzan Young, in preparing for a world trip, converted the following amounts in Canadian currency to the currency of the countries indicated. State the equivalent amounts in the proper currencies.
   (a) $300 England
   (b) $80 Belgium
   (c) $110 Czechoslovakia
   (d) $200 Japan
   (e) $400 Australia
   (f) $55 Brazil
   (g) $90 Mexico
   (h) $84 United States

3. After returning from the world trip Jimmy and Suzan Young sold the following currency to the bank. Calculate the amount they received in Canadian currency for each sale.
   (a) 3.50 English pounds
   (b) 87 Czechoslovakian crowns
   (c) 350 Belgium francs
   (d) 1256 Japanese yens
   (e) 52 Australian dollars
   (f) 26 Brazil cruzeiros
   (g) 124 Mexican pesos
   (h) 32 U.S.A. dollars

4. The cost of an average hotel room for various countries is given below. Which are higher than the Canadian average of $16?
   (a) Scotland, 8.40 pounds
   (b) Austria, 300 schillings
   (c) Belgium, 600 francs
   (d) Denmark, 100 kroners
   (e) Japan, 4800 yen
   (f) Mexico, 160 pesos

14·4 Travelling in Foreign Countries

A wise traveller does not carry large amounts of cash. Usually he carries travellers' cheques, often in U.S. dollars since U.S. dollars are readily accepted in most countries. While Canadian travellers' cheques are available they are not so widely accepted as the U.S. cheques. A Canadian must pay one percent (1 cent per U.S. dollar) to buy U.S. travellers' cheques at any bank. The cheques come in various denominations from $10 to $100. Each cheque is signed by the purchaser at the time he buys it. As he cashes each cheque he signs it again. Hence the signatures can be compared. When the travellers' cheques are cashed the rate of exchange in effect at that time in that country is used. Travellers' cheques are normally insured so if they are lost or stolen, the owner can recover his money.

Example 1

Jose purchased $200 worth of American Travellers' Cheques in Calgary. How much did he pay for them?

Solution

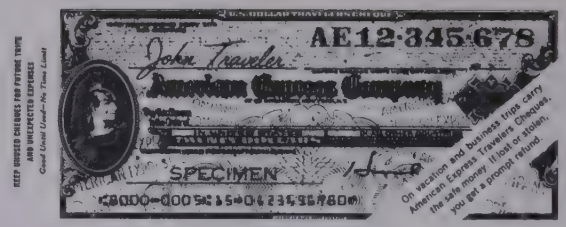
Use the exchange rate on page 239.

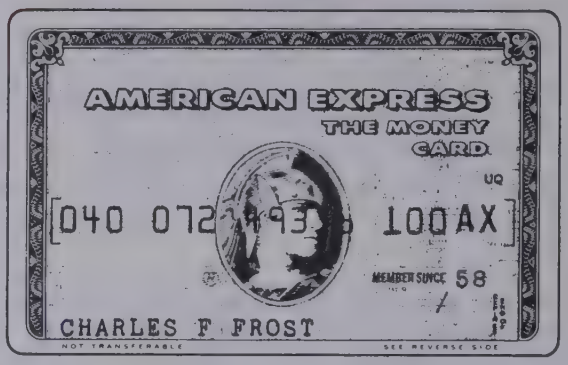
$$\begin{aligned} \$1.00\tfrac{1}{2} \text{ Canadian} &= \$1 \text{ U.S.} \\ \$x \text{ Canadian} &= \$200 \text{ U.S.} \\ x &= 200 \times 1.005 \\ &= 201.00 \end{aligned}$$

The cost of the foreign exchange is $201. There is a 1¢ per U.S. dollar fee for the travellers' cheques.

Total cost is $201.00 + 2.01 = $203.01 Canadian

When a traveller plans to spend a length of time in a particular foreign city, he may obtain a *bank draft* from his home bank *drawn* on its branch or agency in the foreign city. For example, if you planned to spend three months in Rotterdam, Netherlands, you can ask your bank to draw a draft on its agency branch in Rotterdam. When you present the draft to the agency in Rotterdam you will receive guilders even though you paid in Canadian dollars. A draft is a cheque or a bill of exchange.



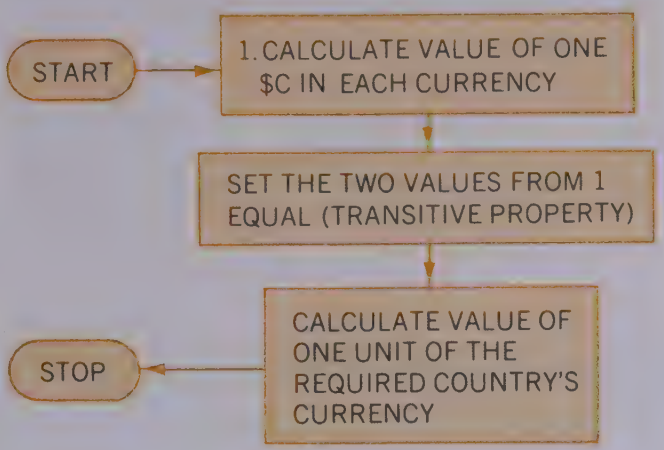


How are American Express, Carte Blanche, Diners' Club, cards (a) different (b) similar to oil company credit cards?

A letter of credit permits the holder to draw any amount of money up to a certain limit specified in the letter, at any of its branches or agencies. There is a maximum and a time limit specified in the letter. An identity card, bearing the owner's signature is presented upon drawing money from the branch or agency.

A number of firms now offer credit cards (American Express, Carte Blanche, Diners' Club, etc.) A person must make application for a card. After the firm has established that the applicant is a good *credit risk* a credit card is issued. There is an annual fee (about $15.00) for the card. Many hotels, stores, airlines, etc., honour the cards. Some companies issuing the credit cards even allow the holder to obtain cash from the company's branches or agencies. If a credit card is lost or stolen the company's office must be notified immediately, in which case the holder of the card is not responsible for any charges made illegally on the card. The holder of the card is billed for all his charges in the currency of his home country. Cards of this type are similar to oil company credit cards.

Occasionally there is need to convert one foreign currency to another given only the rates of each in Canadian funds.


START
1. CALCULATE VALUE OF ONE $C IN EACH CURRENCY
SET THE TWO VALUES FROM 1 EQUAL (TRANSITIVE PROPERTY)
CALCULATE VALUE OF ONE UNIT OF THE REQUIRED COUNTRY'S CURRENCY
STOP

Example 2

What is the value of a British pound in Australian dollars?

Solution

From the table showing the Foreign Exchange Rates at Montreal we get

$1.22 (Canadian) = $1 (Australian)

$2.57 (Canadian) = £1 (British)

$\therefore$ \$1 (Canadian) = $\$\frac{1}{1.22}$ (Australian)

and $\therefore$ \$1 (Canadian) = $£\frac{1}{2.57}$ (British).

Using the transitive property

$£\frac{1}{2.57}$ (British) = $\$\frac{1}{1.22}$ (Australian)

£1 (British) = $\$\frac{1}{1.22} \times 2.57$ (Australian)

£1 (British) = \$2.11 (Australian)

EXERCISE 14·4

Use the exchange rates on page 239 or those from a current newspaper.

1. Ace Bowering of Edmonton pays his bank \$1500 for a bank draft (without charges) on its agency in Tokyo. How many yen did Ace receive in Tokyo?

2. In preparation to travelling in Mexico, Juanita Joseph, purchased in Windsor \$300 U.S. of American travellers' cheques. How much in \$C did she pay?

3. Miss Marcia Wenger drew on a letter of credit 800 francs in Geneva, 600 marks in Berlin and 200 pounds in London. What amount in Canadian funds did she draw?

4. Miss World Traveller presented a bank draft for \$300 (Canadian) in Brussels. How much local money did she receive?

5. Calculate the value:
   (a) of the French franc in German marks
   (b) the Austrian schilling in Danish kroners
   (c) the Yugoslavian dinar in Polish zloty
   (d) the Netherlands guilder in Norwegian kroners.

6. This article appeared in a newspaper.
   (a) What is meant by *non-convertible*?
   (b) What is a kopeck?
   (c) You do not see the foreign exchange rate quoted for the Soviet Union ruble. Why?
   (d) What effect do you think this realignment of the ruble did to the value between the ruble and the Canadian dollar?

RUBLE REALIGNED WITH FOREIGN CURRENCY

MOSCOW -- The Soviet Union has realigned its non-convertible ruble with 22 foreign currencies, lowering the value of all of them except the Japanese yen.

In terms of rubles, the United States dollar fell by eight per cent, the pound sterling by nearly two per cent and the French franc by less than one per cent.

The changes were published by the government newspaper Izvestia, which printed a state bank announcement making the new parities effective immediately.

The changes resembled the broad currency realignment decided at last weekend's Washington meeting of the Group of Ten, the 10 richest non-Communist countries.

Because of the change, one U.S. dollar now buys 82.9 kopecks instead of 90.

Project

Project 1

Purpose

To consider the advisability of purchasing items abroad.

Method

1. Use the library to obtain magazines or travel booklets which show prices of items in foreign countries in foreign currency. For example, the prices of cameras in Germany and Japan can be found in most travel books. These will usually be quoted tax free since you as a tourist would be taking the item out of the country.

   List about $500 worth of items you would like to buy if you were a tourist. Convert the prices to Canadian currency and compare the prices with Canadian prices.

2. Tourists who have been out of Canada for more than 48 hours and less than 1 year are allowed to bring back $100 worth of goods free of duty and sales tax. Any items in excess of this are charged at one of two rates, whichever is to the tourist's advantage:

   (i) the next $100 of value at a flat rate of 25% of fair market value and the balance at the usual duty and sales tax rate.

   (ii) all at the usual duty and sales tax rates.

   Write or obtain from a travel bureau or custom office current regulations for tourists returning from abroad (or use the regulations given above) to calculate what it would cost in taxes and tariff to bring back into Canada the items listed in part 1.

3. Some tourists ask the sales clerk in a foreign store where an item is purchased to make up a bill of sale showing a much reduced price on the item. The sales clerk usually obliges. Discuss why this is done. What are the dangers in this practice as far as the tourist is concerned?

4. If you live near a custom office (railway, airport, or shipping centre) invite a custom official to speak to you. Prepare a set of questions about various aspects of his work.

Using the Library

Erich von Daniken's book *Chariots of the Gods?* published in paperback by Bantam Books, New York 1969, makes one question the greatness of man's scientific advances in the twentieth century. Are they really so outstanding? Should man re-evaluate his view of [illegible] ons and improbable objects in the world which cannot be explained? For instance, mathematically it is incomprehensible that the old stories of the building of the pyramids of Egypt and the stone giants on Easter Island (to name just two gigantic structures) could have been true. (This book was made into a television production and carried on CTV. It may be possible to get a repeat showing.)

REVIEW EXERCISE 14

1. There is a specific tariff of $4.00 per flask of 4 ounces and an ad valorem tariff of $22\frac{1}{2}\%$ on alcoholic perfumes. In addition there is a sales tax of 12%. Calculate the total cost of 1 gross of flasks if the manufacturer's price is $9.20 per flask.

2. A Canadian tourist in Australia saw a boomerang listed at $3.25 (Australian). The exchange rate stated is $1.22 (Canadian) = $1 Australian. What is the price of the boomerang in Canadian dollars?

3. While travelling to Innsbruck, Austria to ski, Nancy Slalom cashed the following U.S. travellers' cheques and spent the money in these cities:

   $300 in Innsbruck $200 in Paris
   $200 in Lucerne $200 in Hanover.

   How much did she spend in each country's currency? Consider the U.S. dollar on par with the Canadian dollar.

Chapter Fifteen

PROBABILITY AND STATISTICS

15·1 Introduction

How many students in your class are 18 years of age?

What part of the whole class is 18 years old?

The above information is relatively easy to obtain. For a start into this study, take a quick poll of your class on the above questions, or on some other topic of information you wish.

The information for the following is not so readily available (and you are not encouraged to "dig out" the answers).

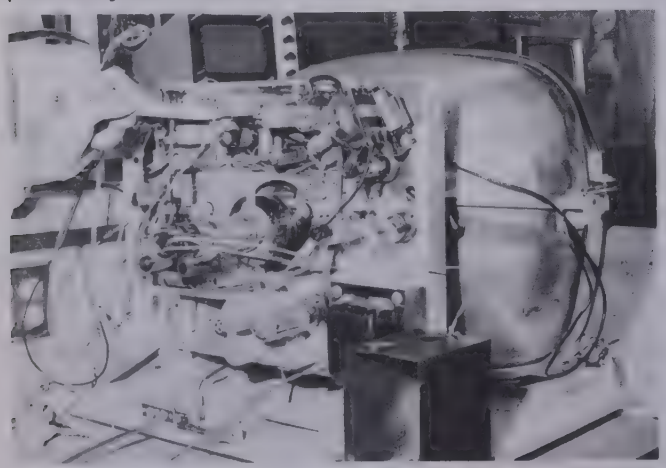
How many 18-year olds are there in the country?

What fraction of the total population are the 18-year olds?

What fraction of the total population are old-age people (70 years of age and over)?

A department of our Federal Government has a record and keeps compiling new records of vital information about our country's population, economic growth, production, employment, etc. Collecting, organizing, analysing, and reading forecasts from such records come under the study called *statistics*.

Statistical records are kept by many organizations and institutions. Manufacturers continually keep testing their products in order to maintain quality control. A manufacturer which produces TV sets would test every finished set, whereas a light bulb manufacturer would sample test its output by checking (for example) every tenth bulb produced.



EXERCISE 15·1

1. As a class project, collect the following data from each student in your class:
   (a) the natural hair color (blonde, brunette, red, black);
   (b) the brand name of toothpaste used by each student.

2. From the data of question 1 answer the following questions.
   (a) What fraction of the total class population are blonde? brunette? etc.
   (b) Is any one toothpaste more popular than the others? Would a manufacturer of toothpaste be interested in data such as this? Why?
   (c) Display your data in some chart or pictorial form.

3. Choose other topics of interest for a class project as suggested in questions 1 and 2.
   Examples are:
   (a) What percentage of students smoke?
   (b) What are the favourite sports of students?
   (c) What are students' plans for post-secondary education (university, college, technical, etc.)?
   (d) Display your data.

15·2 Collecting and Organizing Data

Data are of value when they reveal certain overall information. Before data are collected and organized, one must determine the purpose for such a study. Let us follow through this example and note how data are collected and organized.

Example 1

Bill was interested in finding out where his mathematics test mark of 47 stood in relation to the marks of the rest of the class. He collected the marks from the other students. They were as follows:

46, 67, 51, 63, 58, 78, 37, 60, 54, 66, 51, 58, 54, 72, 63, 40, 48, 59, 67, 82, 43, 64, 47, 74, 52, 57, 67, 53, 62, 58, 50, 61, 55, 54, 51, 59, 47.

He then decided to graph the data. What are the significant steps Bill took in the process?

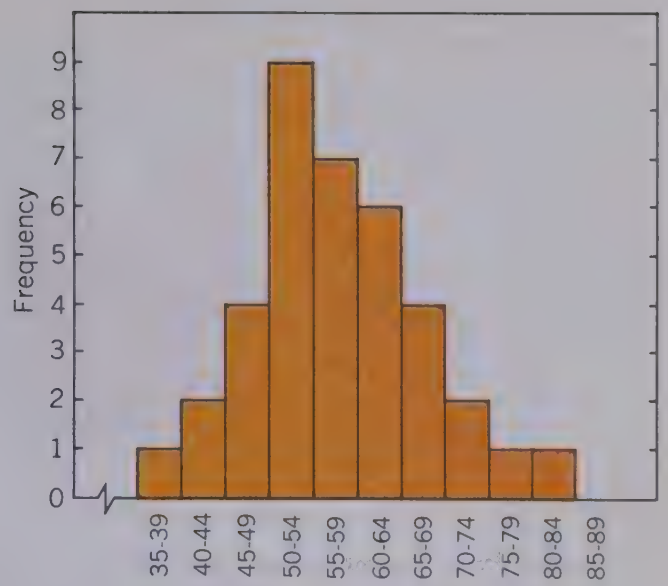
(a) First, Bill organized the data to produce a *frequency table*. He chose the range of five for each *class interval*.

Table 1

Class Interval	*Tally*	*Frequency*
80-84	I	1
75-79	I	1
70-74	II	2
65-69	IIII	4
60-64	𝍸 I	6
55-59	𝍸 II	7
50-54	𝍸 IIII	9
45-49	IIII	4
40-44	II	2
35-39	I	1

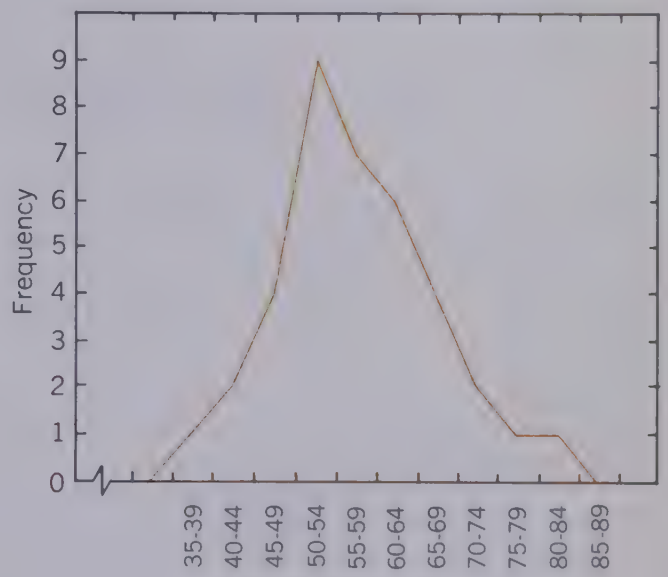
Frequency Table

(b) Then Bill constructed a vertical bar graph from the frequency table. Such a graph is called a *histogram*. The histogram of Figure 15-1 translated all the data from Table 1 into a graphical representation. Another graphical representation of the data in Table 1 could be a *frequency polygon*. Figure 15-2 is such a graph.



Histogram

Figure 15-1

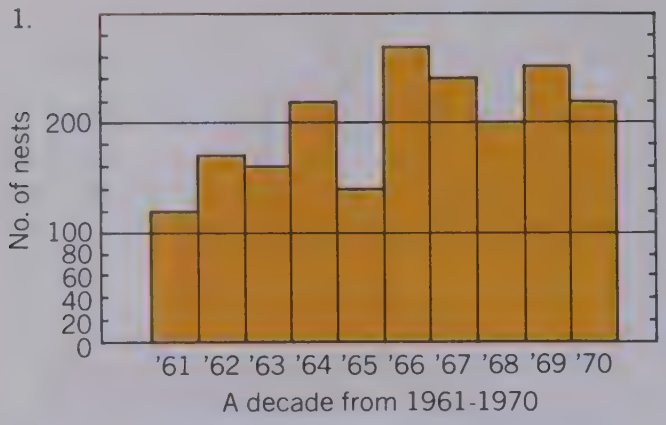


Frequency Polygon

Figure 15-2

EXERCISE 15·2

1.



This graph represents the record of nests counted in Mr. Ecology's Bird Sanctuary for the decade 1961-1970. The count in each year is approximate.

(a) What was the minimum number of nests in the sanctuary for the decade? In what year?

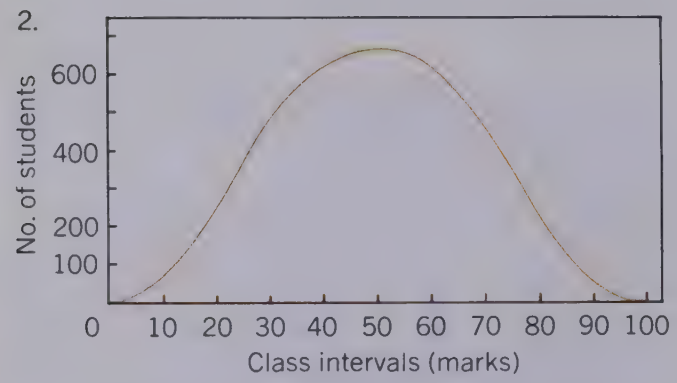
(b) In what year was the approximate number of nests at 140?

(c) How many nests does the graph show for 1962? for 1967?

(d) In what two years were the numbers about the same?

(e) Is there value in this type of record to Mr. Ecology? If so, what value could it serve?

2.



The Grade XII students of Greyland County School Division wrote an aptitude test involving a series of difficult mathematics concepts. There were 100 multiple-choice questions in the test. After the tests were marked the results were represented on a smooth-curve graph.

When the data are plotted so that a smooth curve is fitted to a frequency polygon, the curve is called a *frequency curve*.

(a) Approximately how many students had 20 questions right? 80 questions right?

(b) What was the approximate mark made by 400 students? Did you find two different marks? Explain.

(c) What mark was made by the largest number of students? What is significant about the graph at this mark value? What do we mean by symmetry with respect to this graph?

(d) Name any data that would produce a symmetrical graph such as this.

3. Diane and Rose worked as a team to canvass an area for the United Fund. They called on 40 families. The receipts they issued for amounts received read as follows (all in dollars):

25, 60, 15, 5, 40, 75, 30, 35, 8, 50, 20, 35, 20, 60, 50, 25, 15, 70, 100, 60, 40, 25, 70, 50, 35, 20, 65, 60, 45, 50, 25, 30, 10, 40, 20, 75, 30, 15, 35, 25.

(a) Organize the data into a frequency table (using class intervals 1-10, 11-20, ..., 91-100).

(b) Construct a histogram from the frequency table.

(c) On the histogram in (b) superimpose a frequency polygon for the same data.

4. A survey was taken of the ages of persons who attended a rock music concert. The ages (in years) submitted by the people were as follows:

18, 17, 21, 19, 16, 19, 20, 22, 18, 16, 17, 18, 20, 19, 18, 20, 19, 17, 15, 17, 14, 23, 15, 17, 19, 18, 18, 17, 16, 17, 15, 16, 13, 14, 17, 20, 15, 17, 16, 16, 21, 12, 16, 18, 21, 17, 18, 14, 20, 18, 17, 19, 16, 19, 17, 18, 15, 13, 15, 17, 18, 16, 20, 22, 21, 20, 18, 19, 17, 16, 17, 15, 13, 19, 21, 18, 17, 19, 19, 16, 21, 15, 17, 16, 18, 16, 20, 19, 18, 16, 14, 22, 20, 15, 22, 12, 16, 23, 16, 17, 24, 13, 21, 18, 20, 15, 18, 19, 22, 13, 18, 20, 15, 21, 19, 17, 14, 19, 22, 18, 15, 17, 19, 14, 16, 18, 21, 17, 19, 16, 20, 17, 18, 23, 17.

(a) Organize the data into a frequency table.

Which class interval will you use?
Which age group was the largest in number?

(b) Construct a histogram from the frequency table. What four age groups were the largest number supporters of the concert?

(c) Discuss whether the distribution curve exhibits any symmetry or not.

5. Conduct a class project in dice throwing. Each student throws a pair of dice 20 times and records the sum of the upturned faces for each throw. From each student collect the number of times each sum (i.e., sum of 2, sum of 3, ..., sum of 12) turned up. Organize all the data into a frequency table and then graph it in a histogram.

(a) What sum turned up most often?

(b) What sum(s) turned up least often?

(c) Is the graph symmetrical?

15·3 Mean, Median, Mode

Whether a small number of data or a very large number of data are considered, we are usually curious to know what the average is. A major concern in statistics is with three main kinds of averages for comparison purposes. These three averages are called *mean*, *median* and *mode*.

The *mean*, or *arithmetic mean* as it is sometimes called, is the most common of the three averages. When a student finds his or her average of the report card marks, this average is what statisticians call the mean.

Example 1

Sally wrote a battery of four tests on the topic *Statistics* in her mathematics class. She made marks of 79, 63, 84, 68. What was the mean of these four marks?

Solution

$$\frac{79+63+84+68}{4}=\frac{294}{4}=73.5 \text{ (to one decimal)}$$

When dealing with groups of data where the frequency of certain numbers is greater than one, the operation to find the mean may be simplified. Let us observe the following procedure.

Example 2

In target shooting a perfect score was 10. A dozen boys, out practising on the range, made scores of 7, 9, 8, 8, 6, 8, 7, 5, 7, 9, 8, 6. What was the mean of their scores?

Solution

Especially for a large number of data a frequency table is suggested as the initial approach. The procedure is as follows:

Class Interval	*Tally*	*Frequency*	*Sub-Totals*
5	/	1	5
6	//	2	12
7	///	3	21
8	////	4	32
9	//	2	18
		Total	88

Can you explain how the sub-totals were figured out?

Therefore, the mean is $\frac{88}{12}=7.3$ (one decimal).

Notice: mean $=\dfrac{\text{total of all scores}}{\text{number of scores}}$.

EXERCISE 15·3

1. Lucy's first-term report card showed the following marks in her seven subjects: 76, 89, 63, 65, 82, 71, 77. What was the mean of these marks?

2. A class of students who took extra lessons to improve their knowledge of the quadratic function wrote a multiple-choice test of 40 questions. Their raw scores on this test were: 33, 27, 22, 29, 35, 31, 34, 26, 25, 30, 28, 31, 28, 24.

   Find the mean of these raw scores.

3. Gerald, who became interested in the study of weather, kept various records related to it. Last June he recorded the following temperatures (Fahrenheit) at seven o'clock each morning:
   61, 59, 57, 62, 60, 58, 60, 61, 60, 59, 57, 59, 61, 60, 59, 62, 58, 60, 57, 60, 59, 61, 58, 60, 59, 60, 58, 60, 61, 60.

   What was the mean temperature (at 7 a.m.) for that month of June? *Hint!* First, set up a frequency table as in Example 2 above.

15·4 Median

The median of a set of data is the middle item of the data when it is ordered from the largest to the smallest.

For the following ordered data:

12, 11, 10, 10, 9, 9, 9, 8, 8, 7, 6, 5, 4

the circled nine is the middle item. It is then the median of this set of data.

> The median has as many scores above it as it has below it.

For the following ordered data;

38, 35, 34, 30, 28, 27, 22, 19

there is no middle item. The median for such a set of data is determined by finding the mean of the two middle numbers. The middle numbers in this case are 30 and 28. The median of this set of data is

$$\frac{30+28}{2}=\frac{58}{2}=29.$$

> For an even number of items of the ordered data the median is computed by finding the arithmetic mean of the two middle numbers.

Example 1

The following is a frequency table of sizes of shoes worn by the men working in a downtown garage.

Shoe size	*Frequency*
12	1
10	2
9	13
$8\frac{1}{2}$	8
8	4
$7\frac{1}{2}$	2
$6\frac{1}{2}$	1

What is the median of their shoe sizes?

Solution

First, find the total number of men by summing the frequencies:

$$1+2+13+8+4+2+1=31$$

When the total is an odd number, subtract one from the total and divide by two, i.e.,

$$\frac{31-1}{2}=\frac{30}{2}=15.$$

There are 15 items above and 15 items below the middle item of the ordered data. Therefore, the middle item of the 31 in the set of ordered data is the 16th item (taken from largest to smallest or from smallest to largest).

The 16th item in the ordered data will be found to be size 9. Therefore, the median of the shoe sizes worn by the men in this garage is 9.

Example 2

In the qualifying round for a golf tournament the scores were tabulated into the following table:

Score	*Frequency*
68	1
69	2
70	5
71	7
72	12
73	16
74	7
75	3
76	1

What was the median of the scores?

Solution

The total number of scores is

$$1+2+5+7+12+16+7+3+1=54.$$

When the total is an even number, divide the total by two, i.e.,

$$\tfrac{54}{2}=27.$$

Then, there are 27 items above the median and 27 items below the median in the set of ordered data.

Therefore, the median will be between the 27th and 28th items.

From the data in the above table, the 27th item is the score 72, and the 28th item is the score 73.

Hence, the median of their scores is

$$\frac{72+73}{2}=\frac{145}{2}=72\tfrac{1}{2} \text{ or } 72.5$$

EXERCISE 15·4

1. Ruth was a competitor in the 100-yard hurdles in her school's track and field eliminations. In the heat she ran, the time of each contestant was recorded. The record was as follows (time in seconds): 13.8, 13.9, 14.1, 14.2, 14.5, 14.5, 14.7. What was the median of the running time for this heat?

2. Ralph kept a record of the maximum number of students that were enrolled in his class for each month of the school year. The record showed 33 students in September, 38 in October, 38 in November, 37 in December, 35 in January, 35 in February, 35 in March, 36 in April, 36 in May, 37 in June. What was the median maximum number of students for each month of the school year?

3. During his first ten years in the Bow Valley Hockey League, Rick scored 15 goals in 1961, 22 in 1962, 29 in 1963, 24 in 1964, 31 in 1965, 30 in 1966, 25 in 1967, 35 in 1968, 34 in 1969 and 37 in 1970. What was the median of the goals Rick scored per year?

4. For a biology test of 20 multiple-choice questions the distribution of marks in Cynthia's class was as follows:

Raw Score	*Frequency*
20	1
19	2
18	5
17	3
16	1
15	4
14	3
13	12
12	5

What was the median of the biology marks?

5. Tim and Beth canvassed their church congregation for donations to send handicapped children to summer camp. A tabulation of their receipts showed the following distribution of donations:

Donations	*Frequency*
$ 1	48
$ 2	45
$ 5	42
$ 8	17
$10	8
$15	15
$20	6

What was the median donation contributed?

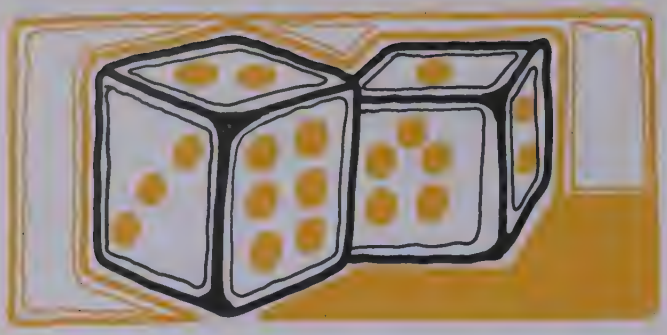
6. What was the median amount contributed to the United Fund in question 3, Exercise 15·2?

7. What was the median age of the people who attended the rock music concert in question 4 of Exercise 15·2?

8. The following heights were cleared in a high jump competition:

Height	*Frequency*
5′4″	3
5′3″	2
5′2″	7
5′1″	13
5′0″	6
4′11″	2
4′10″	5
4′9″	4
4′8″	1
4′7″	4
4′6″	2
4′5″	1

What was the median height?



15·5 Mode

Of the three averages, the mode is the easiest one to determine. The mode is the item which occurs the greatest number of times in a set of data.

Example 1

The marks for a class on a mathematics test of 10 multiple questions are tabulated in the following frequency table.

Class Interval	10	9	8	7	6	5	4
Frequency	2	7	8	12	6	4	3

What is the mode of the above data?

Solution

The frequency table shows the item which appears most frequently as the mark of 7. Therefore, the mode of this data is 7.

EXERCISE 15·5

1. Find the mode for the data in
   (a) *Table 1* of this chapter
   (b) question 2, Exercise 15·2
   (c) Example 2, Section 15·4.

2. What is the mode for the data in this chapter for
   (a) Exercise 15·2, question 3?
   (b) Exercise 15·2, question 4?
   (c) Exercise 15·4, question 4?
   (d) Exercise 15·4, question 5?

Project

Project 1

Purpose

To determine the role of statistics in our daily lives.

Method

1. Each student collects statistical information from recent issues of newspapers, magazines, periodicals, advertising posters and from radio and television broadcasts.
2. Bring all the information to class for analysis and evaluation.
3. Arrange into categories before having a total class (or small groups) discussion.

Questions for Discussion

1. What is the intent of each illustration?
2. Which are most easily interpreted? Which are the hardest to interpret?
3. Are any of the illustrations misleading? If so, in what way?

15·6 Probability

How would you respond to this question? "If a ball is thrown up into the air, what are the chances that it will come down?"

Your reply may be, "There is a 100% chance that it will come down." Or you may say, "I am positive that it will come down."

On the other hand, what might be your response to "What are the chances of a fine, warm, sunny day for our outing next Saturday?"

A definite reply like "I'm sure it will be a fine day," cannot be taken in the same light as the reply above. "The chances are in favour of a good day," or "It will probably be a fine day" are reactions more in keeping with reality.

Statements like "It will probably be a fine day," are used quite frequently without our realizing that we are expressing a probability. In the history of mathematics the study of probability is quite recent, dating to the seventeenth century for its beginning. (See *Using the Library*.) It wasn't until this century that the laws of probability have been seriously applied in industry, business, medicine and science. One of the earliest applications of probability was made in the field of insurance. Policy premiums for life insurance are based on the probability that on the average the insured will live to an expected age. Government taxation is based on the probability that the economy will continue growing at a certain rate.

Empirical Probability

When data are obtained from real experiences or from experiments, and probability figures are based on the data, we define these probability figures as empirical probabilities.

Tables have been developed which indicate the average number of years a person of a given age will live. These tables are called *Life Expectancy Tables* or *Mortality Tables*. These tables used by life insurance companies are the results of experiences with large numbers of cases over a long period of time. This kind of statistical data yields statistical probability figures which are, by definition, empirical probability figures.

We speak about the *probability* some event will happen. On the basis of empirical probability, we mean the ratio of the number of times the particular event A happened to the total number of events B. We write

$$P(\text{event } A) = \frac{\text{number of times event } A \text{ happened}}{\text{number of total events } B}$$

P(event A) means *probability of event A*.

What is the relationship between odds and probability?

Example 1

John experimented with his pet mouse to find out what its reaction would be to color. In a cage he set up a divider with three colored doors, red, yellow and blue which he could change at will. On the other side of the doors he planted a piece of cheese as a reward for each trial. He collected data on his experiment over a period of 30 days. The data revealed that the mouse chose to go through the red door 20 times, through the yellow door 36 times and through the blue door 4 times. From this evidence what was the probability that the mouse, on its next trial, would go through (a) the red door, (b) the yellow door, (c) the blue door?

Find the sum of these three probabilities. What do you notice?

Solution

Total number of trials = 20 + 36 + 4 = 60.

(i) P(red door)

$$= \frac{\text{number of times mouse chose red door}}{\text{no. of times he chose all doors (no. of trials)}}$$

$P(\text{red door}) = \frac{20}{60} = \frac{1}{3}$

(ii) P(yellow door)

$$= \frac{\text{number of times mouse chose yellow door}}{\text{total number of trials}}$$

$P(\text{yellow door}) = \frac{36}{60} = \frac{3}{5}$

(iii) P(blue door)

$$= \frac{\text{number of times mouse chose blue door}}{\text{total number of trials}}$$

$P(\text{blue door}) = \frac{4}{60} = \frac{1}{15}$

Sum: P(red door) + P(yellow door) + P(blue door)

$= \frac{1}{3} + \frac{3}{5} + \frac{1}{15} = \frac{15}{15} = 1.$

The sum of all the probabilities is 1.

EXERCISE 15·6

1. A grocer observed from his records that in the past year his customers bought 1200 lb of brand A cheese and 800 lb of brand B cheese. What is the probability that the next brand of cheese bought by a customer will be brand B?

2. The enrolment figures in Leeds High School show 330 students in Grade XII, 270 students in Grade XI and 210 students in Grade X. What is the probability that a student of Leeds High, chosen at random from an assembly, is a Grade XI student?

3. A loaded die (one that is weighted to one side) was found to have an empirical probability of $\frac{1}{4}$ for the number 5 coming up. If so, how many times should number 5 come up in the next dozen throws?

4. An automobile dealer observed that customers in the past four months bought 68 white cars, 96 yellow cars, 44 red cars and 32 other colored cars. What is the probability that the next customer will buy a yellow car? Can we determine the probability that the next car sold will be a green one? Why?

5. Determine empirically the probability that any one person in your class is
   (a) 18 years old (b) 17 years old
   (c) over 18 years old (d) under 17 years old.
   What is the sum of the four probabilities? Can you explain this sum?

6. Determine empirically the probability that any one person in your class
   (a) has blue eyes
   (b) has brown eyes
   (c) has green eyes.
   Does the sum of these three probabilities equal 1? Why?

7. Determine empirically the probability that a student in your class
   (a) has a valid driver's licence
   (b) has a learner's permit to drive
   (c) has no driver's licence or permit.
   What is the sum of these three probabilities? Explain your sum.

Projects

Project 2

Labs to Determine Empirical Probabilities

A. *Purpose*

To find the probability that a falling thumbtack will land pin up.

Method

1. Take 10 thumbtacks of equal-sized pin and head.
2. From a height of six to ten inches let them fall on a level surface (table top, floor).
3. Count the number that landed pin up and those that didn't.
4. Repeat part 3 for ten or more times.
5. Add up the total tally for those landing pin up.
6. Find the total tally of the number of tacks tossed.
7. Find the empirical probability of a falling thumbtack landing pin up.
8. Will the probability be the same for other sized thumbtacks? Repeat the experiment using other sized thumbtacks and compare the probabilities.

B. *Purpose*

To find the probability that a person from your class was born in June, July or August.

Method

1. Record the month of birth for each person in your class.
2. Find the total of those who were born in June, July or August.
3. Find the empirical probability that a person from your class was born in June, July or August.
4. Find other probabilities from the data collected.
5. Try to obtain similar data from another class and compare probabilities.

C. Devise other lab activities to find the empirical probabilities of happenings in your own surroundings and localities.

Project 3

Purpose

To determine the implications of the *Law of Large Numbers.*

Method

1. Each student in class supplies himself with a one-cent coin, pencil and paper (for tabulating).
2. Each student tosses the coin ten times, tossing it the same way each time.
3. Record the number of heads and the number of tails.
4. On the basis of the ten tosses calculate the empirical probability of heads coming up.
5. Compare the probabilities other students computed.
6. Each student tallies the heads or tails for 100 tosses. After the 100 tosses for each student are completed tabulate the total heads and total tails for the whole class.

Questions

1. List all the students' empirical probabilities for heads in the first ten tosses. Observe the wide variations of these probabilities. How many students reported probabilities greater than one half? How many reported less than one half?
2. What was the number of total tosses for the class (each student contributing 100)? Of this total how many were heads? tails?
   What is significant about the totals of heads and tails?
3. Compare the individual probabilities (for heads) from the first ten tosses with the probability (for heads) for the class total. What would we expect the probability (for heads) to be for another class total experiment with the same number of tosses?
4. What are the implications (on probability) of experiments where very large numbers of repeated trials are used?

15·7 Probability by Reasoning

A fair (not weighted) coin is tossed. What is the probability that it will come up heads?

This is one of the most elementary problems in probability. To answer this question one must reason out the range of possible values, sometimes called the *possible outcomes*. Since the coin can come up either heads or tails, there are just two possible outcomes. In dealing with this kind of event it is important to know that each is an equally likely outcome – hence a "fair coin" must be used. Secondly, one must determine how many of the range of possible values are favourable to the event "come up heads". Because of the two possible outcomes, heads can come up once, then the

$$P(\text{a head}) = \frac{\text{number of ways a head can come up}}{\text{total ways coin can land}}$$

$$P(\text{a head}) = \tfrac{1}{2}.$$

Let us generalize by letting

p = the range of possible outcomes
f = the favourable outcomes corresponding to a specific event.
$P(A)$ = the probability of the event A

We can now write

$$P(A) = \frac{f}{p} = \frac{\text{favourable outcomes}}{\text{possible outcomes}}.$$

Example 1

Cards, numbered 1, 2, 3, 4, 5, are thrown into a box. What is the probability that if one card is drawn it will be an even-numbered card?

Solution

Since there are 5 different cards that can be drawn, the total number of possible outcomes for this event is $p = 5$. Of these outcomes, the favourable number is $f = 2$ (since there are only 2 even-numbered cards). Therefore, the probability of this event is 2 out of 5, or

$$P(E) = \frac{f}{p} = \frac{2}{5}.$$

Example 2

A letter is chosen at random from the word *Mississippi*. What is the probability that the letter chosen is
(a) a consonant?
(b) the letter s?
(c) the letter *p*?

Solution

(a) Since there are 7 consonants and 11 letters altogether we get

$$P(\text{a consonant}) = \frac{f}{p} = \frac{7}{11}$$

(b) Since there are 4 s's and 11 letters altogether we get

$$P(s) = \frac{f}{p} = \frac{4}{11}$$

(c) Since there are only 2 p's we get

$$P(p) = \frac{2}{11}$$

Do questions 1 through 6 in Exercise 15·7.

Example 3

Two cards are drawn from a deck of 52 playing cards. What is the probability of drawing a sum of "three" for the two cards?

Solution

There are 52 ways of drawing the first of the two cards. For every possible first choice, there are 51 choices for the second card (since there are 51 cards left for the second draw). Therefore, there are 52×51 ways of drawing 2 cards. Hence, $p = 52 \times 51$.

The favourable outcomes of the event "a sum of three" is equal to 4×4. To make "a sum of three" the two cards must be an ace and a two. For every ace there are 4 two's. Because there are four aces then $f = 4 \times 4$. Therefore,

$$P(E) = \frac{f}{p} = \frac{4 \times 4}{52 \times 51} = \frac{4}{663}.$$

Example 4

From a class of 18 girls and 15 boys, two are chosen as representatives to the Students' Council. What is the probability that both are girls?

Solution

The possible ways of choosing 2 representatives from 33 students are 33×32, i.e., 33 persons from which the first choice is made and 32 persons from which the second choice is made.

The favourable outcomes of the event "both are girls" is equal to 18×17, i.e., since the 2 representatives are to be girls there are 18 from which to make the first choice and then 17 for the second choice. Therefore,

$$P(2 \text{ girls}) = \frac{f}{p} = \frac{18 \times 17}{33 \times 32} = \frac{51}{176}.$$

EXERCISE 15·7

1. What do we mean when we say "a fair coin (not weighted)"?

2. A die is tossed. What is the probability that it shows:
   (a) a six? (b) less than a three?
   (c) an odd number? (d) greater than one?

3. A letter is chosen at random from the word LULLABY. What is the probability that the letter is an L?

4. A card is chosen from a deck of 52 playing cards. What is the probability that the card chosen is
   (a) an ace? (b) a face card?
   (c) a club? (d) red?

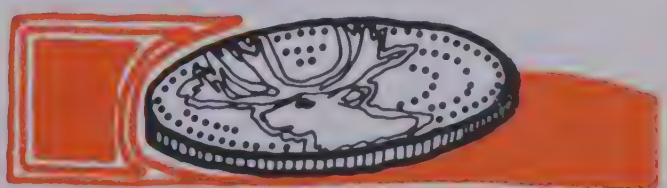
5. In a bag are 4 white marbles and 3 blue marbles. What is the probability that if 1 ball is drawn it is blue?

6. A box contains 2 red balls, 5 yellow balls, and 3 blue balls. What is the probability that a ball chosen at random is (a) red? (b) yellow?

7. From a committee of 8 girls and 4 boys two persons are chosen as chairman and secretary. What is the probability that both are (a) boys? (b) girls?

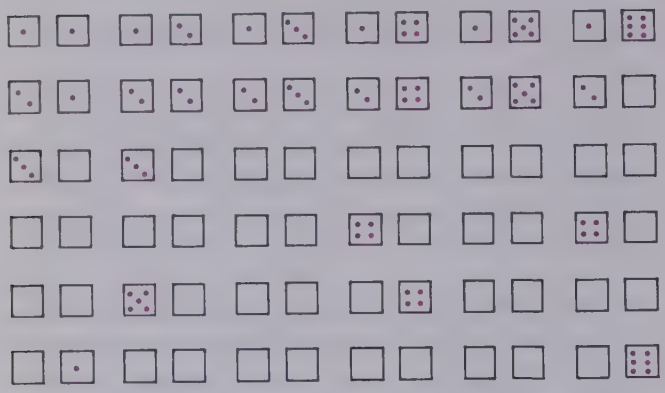
8. Two indistinguishable coins are tossed simultaneously.
   (a) List all the possible outcomes.
   (b) What is the probability that one coin will show "heads" and the other "tails"?

9. Three indistinguishable coins are tossed simultaneously.
   (a) List all the possible outcomes.
   (b) What is the probability that all three show the same?
   (c) What is the probability that two show "heads" and one shows "tails"?



10. Two dice, one red and one black, are tossed. Complete the following table of ordered pairs to show all the possible outcomes of upturned faces.

This chart shows the possible throws.



(1,1)	(1,2)	(1,3)	(1,4)	( )	( )
(2,1)	(2,2)	( )	( )	( )	( )
(3,1)	( )	( )	( )	( )	( )
( )	( )	( )	(4,4)	( )	( )
( )	( )	( )	( )	( )	(5,6)
( )	( )	(6,3)	( )	( )	( )

(a) What is the total of possible outcomes?

(b) How many ordered pairs show a total of 11? What is the probability of obtaining a total of "eleven" on tossing two dice?

(c) What is the probability of obtaining a total of "seven"?

(d) What is the probability of obtaining a total of "less than five"?

(e) What is the probability of obtaining a pair?

(f) What is the probability of obtaining one die showing double of the other?

11. Two cards are drawn from a deck of 52 cards.

(a) What is the probability that both will be black?

(b) What is the probability that both will be kings?

(c) What is the probability that one will be red and the other black?

(d) What is the probability that both will be face cards?

12. Five cards are dealt from a deck of 52 playing cards. Determine the probability of

(a) five cards of the same suit; (b) four aces.

13. Five boys are eligible to win two tickets to the Stanley Cup Finals. Bill and Jack are close friends and would like to go together. What is the probability that Bill and Jack will be chosen?

Hint! There is only one favourable outcome, i.e., only one way to choose Bill and Jack together.

14. (a) Obtain several paper cups of the same kind. Determine empirically the probability that a cup will land right-side-up when pushed fairly off a table edge.

(b) Does the shape of the cup have any relation to the probability?

15·8 Use of Tree Diagrams

Three coins are tossed, either singly and in succession or all at once. To list all the possible outcomes a tree diagram is very useful.

Let us designate the coins as *first coin*, *second coin* and *third coin*. The *first coin* can come up either heads or tails (H or T). For either a head or tail of the *first coin* the *second coin* can come up heads or tails. Similarly, for either a head or tail of the *second coin* the *third coin* can come up heads or tails. The following tree diagram illustrates this and combines them for the list of possible outcomes.

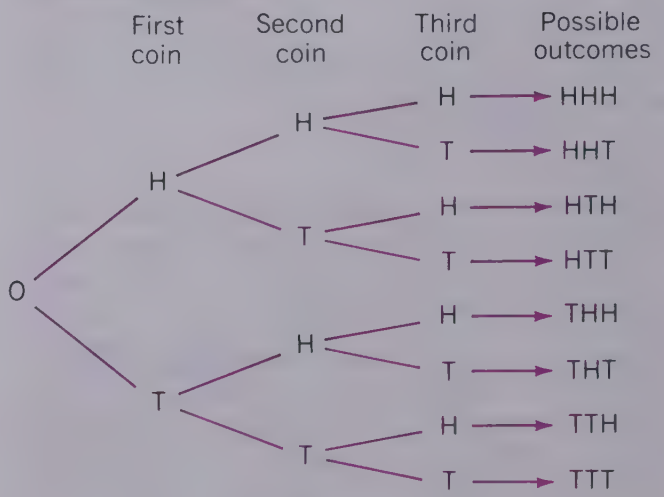


Figure 15-3

Example 1

Refer to Figure 15-3.

(a) How many possible outcomes are there to tossing three coins?

(b) What is the probability of obtaining
(i) three heads?
(ii) two heads and one tail?
(iii) at least one head?

Solution

(a) The total possible outcomes is 8.
Therefore, $p = 8$

(b) For favourable outcomes:
(i) three heads, $f = 1$ (HHH).
(ii) two heads and one tail,
$f = 3$[(HHT)(HTH)(THH)].
(iii) at least one head,
$f = 7$[(HHH)(HHT)(HTH)(HTT)(THH)(THT)(TTH)]

Therefore, the probabilities are

(i) $P(3 \text{ heads}) = \frac{f}{p} = \frac{1}{8}$

(ii) $P(2 \text{ heads, } 1 \text{ tail}) = \frac{f}{p} = \frac{3}{8}$

(iii) $P(\text{at least 1 head}) = \frac{f}{p} = \frac{7}{8}$.

EXERCISE 15·8

1. Make a tree diagram to show the possible outcomes when four coins are tossed.
   (a) How many possible outcomes are there when four coins are tossed?
   (b) Use the tree diagram to determine the favourable outcomes for the following:
      (i) two heads and two tails;
      (ii) one head and three tails;
      (iii) all tails;
      (iv) at least two tails.
   (c) Use the tree diagram to determine the probabilities of:
      (i) three heads and one tail;
      (ii) no tails;
      (iii) at least one head.

2. Use a tree diagram to determine the possible outcomes of having three children in the family as to their sex and position regarding first, second and third born.
   (a) What is the probability that the first two children will be boys?
   (b) What is the probability that the first child is a girl?
   (c) What is the probability that the last two children are girls?
   (d) What is the probability that the second child is a boy?

3. In a four-children family what are the following probabilities:
   (a) three girls and one boy?
   (b) at least two girls?
   (c) the first and last are boys?
   (d) the last two are girls?
   (e) first two are girls and last two are boys?
   (f) two boys and two girls?
   (g) at least three boys?

15·9 Probabilities of Compound Events: P(A or B), P(A and B)

Intersection of Sets

Let set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 4, 5, 6\}$.

Then, the set of elements $\{3, 5\}$ is common to both sets and we say that $\{3, 5\}$ is found in A and B. The notation we use for *A and B* is $A \cap B$ and we read it *A intersect B*. Graphically we represent the intersection of sets A and B in Figure 15-4 by the shaded part.

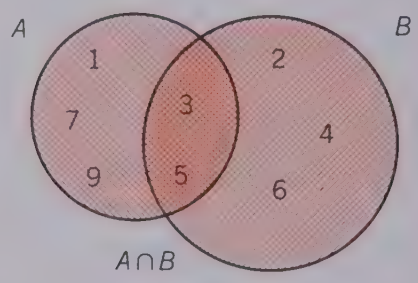


Figure 15-4

When two sets do not have a member in common the intersection of the two sets is the empty or null set. This is indicated $A \cap B = \phi$.

Union of Sets

Let us use the same sets A and B as above. There are eight elements in A or B. The notation we use for *A or B* is $A \cup B$ and we read it *A union B*. Graphically we represent the union of sets A and B in Figure 15-5 by the shaded part.

Therefore, $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$.

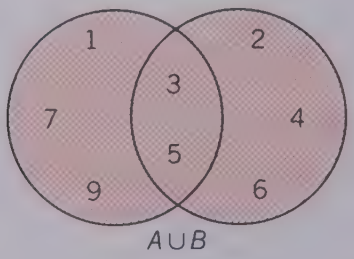


Figure 15-5

Project

Project 4

Purpose

To study the probabilities of two events (occurring independently of each other) as *P*(union of the two events) and *P*(intersection of the two events).

Method

A. 1. Each student provides himself with a supply of large-squared graph paper, pencil and ruler.

2. On the graph paper section off a portion of the first quadrant as in Figure 1. Heavily dot the points representing the ordered pairs (x,y) such that $1 \le x \le 5$ and $1 \le y \le 5$.

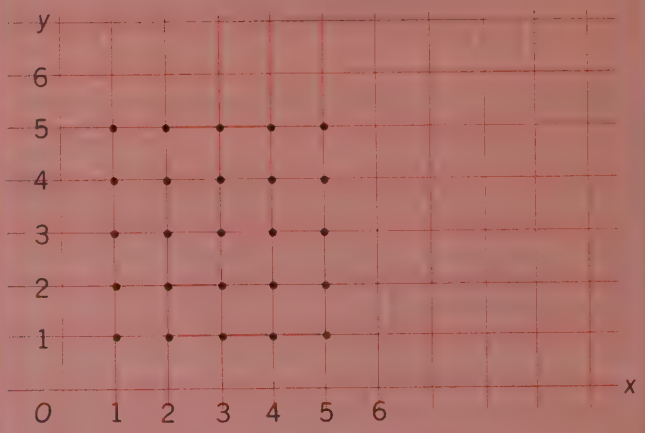


Figure 1

3. Let A be the event "$1 < x < 5$" and B be the event "$3 \le y \le 5$".

4. Determine: $P(A)$, $P(B)$;
$P(A \text{ or } B)$, i.e., $P(A \cup B)$;
$P(A \text{ and } B)$, i.e., $P(A \cap B)$.

B. 1. On another part of the graph paper section off a portion of the first quadrant with dots representing the ordered pairs (x,y) such that $1 \le x \le 8$ and $1 \le y \le 8$.

2. Let H be the event *sum of ordered pair is greater than 7.*

Let K be the event $y > 4$.

3. Determine $P(H)$, $P(K)$, $P(H \cup K)$, $P(H \cap K)$.

C. 1. On another part of the graph paper, section off a portion of the first quadrant as in Figure 2. Heavily dot the points as in Figure 2 to represent the ordered pairs in that first quadrant.

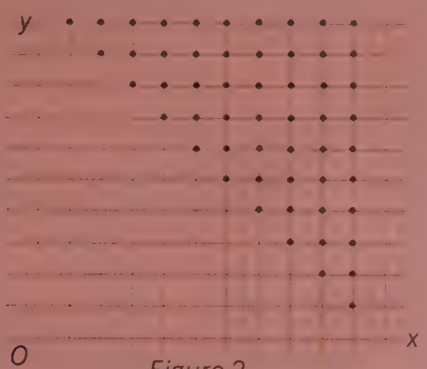


Figure 2

2. Let R be the event $y > 6$.
Let S be the event $x + y > 14$.

3. Determine $P(R)$, $P(S)$, $P(R \cup S)$, $P(R \cap S)$.

D. 1. On another part of graph paper section off a portion of the first quadrant. Heavily dot the points representing the ordered pairs (x,y) such that $4 \leq x \leq 10$ and $1 \leq y \leq 8$.

2. Let M be the event $x + y < 9$.
Let N be the event $x + y = 9$.

3. Determine $P(M)$, $P(N)$, $P(M \cup N)$, $P(M \cap N)$.

E. 1. Consider the set of lattice points in Figure 3. The total number of points is set T; and subsets A, B and C are marked off as shown in Figure 3. Many life situations can be represented by a Venn diagram such as this.

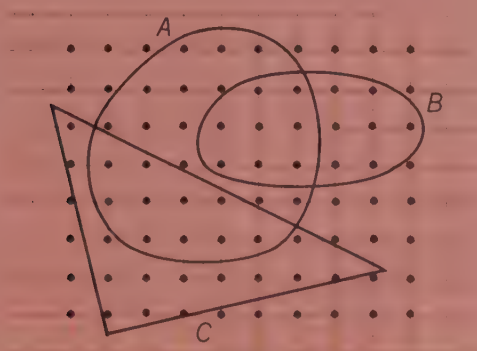


Figure 3

Suppose T represents the set of all boys in Grade XII, A represents the subset of boys who are 6 feet tall or taller, B represents the subset of boys who are fair-haired and C represents the subset of boys who are dark-haired.

2. Let
T be the event all boys in Grade XII.
A be the event all boys 6 feet or taller in Grade XII.
B be the event all fair-haired boys in Grade XII.
C be the event all dark-haired boys in Grade XII.

3. Determine $P(A)$, $P(B)$, $P(C)$, $P(A \cup B)$, $P(A \cup C)$, $P(B \cup C)$, $P(A \cup B \cup C)$, $P[T - (A \cup B \cup C)]$.

Questions

1. On Figure 4 draw in the boundaries to represent:
Event $A(1 < x < 5)$
Event $B(3 \leq y \leq 5)$.
Figure 4 is the result of the drawings. This is a Venn diagram of the universal set, and subsets A and B.

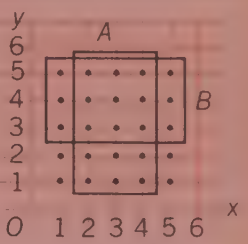


Figure 4

From the diagram read:
number of elements in universal set _____;
number of elements in event A _____;
number of elements in event B _____.

Determine:

$$P(A) = \frac{\text{number in event } A}{\text{number in universe}} = \_\_\_\_\_$$

$$P(B) = \frac{\text{number in event } B}{\text{number in universe}} = \_\_\_\_\_$$

$$P(A \cup B) = \frac{\text{number in } A \cup B}{\text{number in universe}} = \_\_\_\_\_$$

$$P(A \cap B) = \frac{\text{number in } A \cap B}{\text{number in universe}} = \_\_\_\_\_$$

2. Proceed in same manner as in question 1 above for the figures drawn in parts B, C and D. Answer all the questions for each part that you answered in question 1.

3. Express $P(A \cup B)$ in terms of $P(A)$, $P(B)$, $P(A \cap B)$, when A and B are events of some outcome.
Does the expression hold true for both cases, when $A \cap B \neq \phi$ and $A \cap B = \phi$?

Example 1

Of a total sample of 1500 males canvassed, 800 were married, 300 were from 20 to 30 years old and 100 were married from 20 to 30 years old. If a male (from this group) were chosen at random what is the probability that he

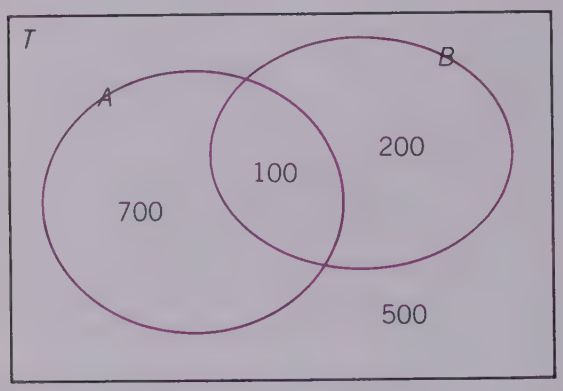
(a) was from 20 to 30 years old and single?

(b) was not married and was not from 20 to 30 years old?

(c) was married and was not from 20 to 30 years old?

(d) was married or from 20 to 30 years old?

Solution



Let A be the event males married. (800)

Let B be the event males from 20 to 30 years old. (300)

Let T be the total number of males canvassed. (1500)

Use a Venn diagram to determine the number of males in

$$A - B = 700$$
$$B - A = 200$$
$$A \cup B = 1000$$
$$T - (A \cup B) = 500$$

(a) P(20 to 30 years old and single) $= \frac{200}{1500} = \frac{2}{15}$

(b) P(not married and not 20 to 30 years old) $= \frac{500}{1500} = \frac{1}{3}$

(c) P(married and not 20 to 30 years old) $= \frac{700}{1500} = \frac{7}{15}$

(d) P(married or 20 to 30 years old) $= \frac{1000}{1500} = \frac{2}{3}$

EXERCISE 15·9

1. Consider a two dice experiment, one red and the other white. With red along the x-axis and white along the y-axis, set up the points in the first quadrant corresponding to all the possible outcomes for the two dice.
   (a) How many ordered pairs represent the possible outcomes?
   (b) How many ordered pairs show red equals white?

2. For the following use the set of points from question 1. If two dice are rolled (red and white) what is the probability:
   (a) that red die shows 3 or less and white die shows 5 or more?
   (b) $r = w$ (where r stands for red, w for white)?
   (c) $r + w = 7$?
   (d) $r + w = 11$?
   (e) $r = 1$ and $w = 1$?
   (f) $r = 1$ or $w = 1$?
   (g) $r + w > 6$ and $w > 3$?
   (h) $w \neq 6$ and $r + w = 8$?
   (i) $r < 3$ or $w > 3$?
   (j) $r + w > 4$ or $w = 5$?

3. In the last semester examinations, 18% of the students failed mathematics, 12% failed English and 5% failed English and mathematics. If a student who wrote both examinations is chosen at random what is the probability that:
   (a) he failed both examinations?
   (b) he failed English and not mathematics?
   (c) he passed both examinations?

4. For the experiment, Tossing Three Coins, list all the possible ordered triples. (See Figure 15-3.)
   Let event A = at least two heads.
   Let event B = at least one tail
   for the set of ordered triples.
   Determine the following probabilities:
   (i) $P(A)$ (ii) $P(B)$ (iii) $P(A \cup B)$ (iv) $P(A \cap B)$

5. A card is chosen at random from a deck of 52 cards.
   Let event F = {choosing a face card (J, Q, K)}.
   Let event R = {choosing a red card}.
   Determine the following probabilities:
   (i) $P(F)$ (ii) $P(R)$ (iii) $P(F \cup R)$ (iv) $P(F \cap R)$.

6. One hundred students were canvassed. 37 reported that they were blue-eyed, 38 were blonde and 15 were blonde and blue-eyed. What is the probability that one person (of the 100) chosen at random,
   (i) is blonde and blue-eyed?
   (ii) is blonde or blue-eyed?
   (iii) is neither blonde nor blue-eyed?
   (iv) is blonde but not blue-eyed?
   (v) is blue-eyed but not blonde?

7. In a group of 30 students, 20 are enrolled in mathematics or mathematics and physics; 17 are enrolled in physics or physics and mathematics. If a student is chosen at random from the group, what is the probability that he is enrolled in,
   (i) mathematics and physics?
   (ii) mathematics only?
   (iii) physics only?

8. A survey of 900 adults revealed that 450 had to have glasses prescribed to improve vision, 160 had glasses prescribed and had tooth extractions, 270 had tooth extractions but no glasses prescribed. If this survey is representative of our total population, what is the probability that an adult chosen at random,
   (i) wears prescribed glasses?
   (ii) had a tooth extraction?
   (iii) wears prescribed glasses or had a tooth extraction?
   (iv) does not wear glasses nor had a tooth extraction?

Using the Library

1. A book that deals with mathematics only in an indirect way is Ralph Nader's book, *Unsafe at Any Speed*, published by Pocket Books, a division of Simon and Schuster, New York. Here is a book by a man who works with a team of researchers who collect and/or interpret statistics in such a way as to influence the lives of most North Americans and even people in other parts of the world. This book deals with Nader's attack on the automobile industry. Like many books you can read those chapters that most interest you – though once you have started reading you will likely want to read the whole book.

2. Pascal and Fermat developed the theory of probability when asked by a gambler who was losing on a certain combination in the roll of 3 dice. Research some of their early theories.

3. Many people have related numbers and mysticism in numerology. Find out what you can about this. GEMATRIA may give you a clue.

4. Read *The World of Mathematics* pages 1449-1451. Jakob Bernoulli stated a theorem in probability later known as Bernoulli's Theorem. Simeon Poisson was also a great teacher in probability. Write a brief comment on a false way of using the Law of Averages in gambling. State in your own words wherein lies the misinterpretation of the Law of Averages as made by the ill-informed.

5. Read for interest only, *How to Lie with Statistics* by Huff.

6. Through the study of vital statistics, John Graunt attempted to prove or disprove certain common arguments of the day. Some of these arguments, current thoughts of the mid 1600's may seem strange and interesting to us today.

   Read pages 1421-1435 of *The World of Mathematics*.

   List those arguments that John Graunt, a founder of vital statistics, attempted to support.

7. Read *The World of Mathematics*, pages 1524 ff. In the gambling world what does "Two to one bar one" mean?

   George Bernard Shaw makes the statement, "An insurance company, sanely directed, and making scores of thousands of bets, is not gambling at all." In what way might one think an insurance company gambles, and in what way is G. B. Shaw correct?

REVIEW EXERCISE 15

1. The Fish and Game League held a fish derby for junior boys. Every fish that was caught was weighed in ounces. The weights were as follows:

   15, 21, 18, 17, 14, 20, 18, 19, 15, 19, 17, 18, 16, 22, 17, 14, 19, 18, 21, 17, 18, 13, 16, 18, 19, 18, 16, 20, 17, 18, 15, 19, 21, 16, 18, 19, 18, 20, 19, 16, 17, 18, 15, 20, 19, 17, 18, 18, 20, 15, 16, 19, 17, 18, 12, 17, 19, 21, 18, 14, 19, 18, 19, 20, 18, 16, 14, 20, 18, 19, 18, 20

   (a) Organize the data into a frequency table.

   (b) Construct a histogram from the frequency table.

   (c) Determine the mean, median and mode of the distribution.

2. The following is a frequency table of the ages of residents of Kunawa Village.

Class Interval	*Frequency*
70 - 79	6
60 - 69	24
50 - 59	33
40 - 49	30
30 - 39	38
20 - 29	27
10 - 19	15
0 - 9	13

   For the above data determine the mean, median and mode.

3. A card is drawn at random from a deck of 52 cards. What is the probability that the card is,

   (a) a red jack, queen, or king?

   (b) a black card less than an eight?

   (c) a four or a five?

   (d) an ace or a king?

   (e) a club greater than an ace?

4. In a bag are 8 red marbles, 11 green marbles, and 13 white marbles. What is the probability that when one marble is drawn that it is red? What is the probability that when two marbles are drawn that they are both white?

5. The Winterclub Membership included 240 ladies. 112 ladies were in the curling league, 128 ladies were in the badminton league, and 32 ladies were in both leagues (curling and badminton). If a lady member of the Winterclub were chosen at random, what is the probability that she was,

   (a) not in the curling or badminton league?

   (b) in the badminton or the curling league?

   (c) in the curling league and the badminton league?

   (d) in the curling league but not in the badminton league?

   (e) in the badminton league but not in the curling league?

TABLE OF SINES, COSINES, TANGENTS

Angle (Degrees)	Sine	Cosine	Tangent
0°	.0000	1.0000	.0000
1°	.0175	.9998	.0175
2°	.0349	.9994	.0349
3°	.0523	.9986	.0524
4°	.0698	.9976	.0699
5°	.0872	.9962	.0875
6°	.1045	.9945	.1051
7°	.1219	.9925	.1228
8°	.1392	.9903	.1405
9°	.1564	.9877	.1584
10°	.1736	.9848	.1763
11°	.1908	.9816	.1944
12°	.2079	.9781	.2126
13°	.2250	.9744	.2309
14°	.2419	.9703	.2493
15°	.2588	.9659	.2679
16°	.2756	.9613	.2867
17°	.2924	.9563	.3057
18°	.3090	.9511	.3249
19°	.3256	.9455	.3443
20°	.3420	.9397	.3640
21°	.3584	.9336	.3839
22°	.3746	.9272	.4040
23°	.3907	.9205	.4245
24°	.4067	.9135	.4452
25°	.4226	.9063	.4663
26°	.4384	.8988	.4877
27°	.4540	.8910	.5095
28°	.4695	.8829	.5317
29°	.4848	.8746	.5543
30°	.5000	.8660	.5774
31°	.5150	.8572	.6009
32°	.5299	.8480	.6249
33°	.5446	.8387	.6494
34°	.5592	.8290	.6745
35°	.5736	.8192	.7002
36°	.5878	.8090	.7265
37°	.6018	.7986	.7536
38°	.6157	.7880	.7813
39°	.6293	.7771	.8098
40°	.6428	.7660	.8391
41°	.6561	.7547	.8693
42°	.6691	.7431	.9004
43°	.6820	.7314	.9325
44°	.6947	.7193	.9657
45°	.7071	.7071	1.0000

Angle (Degrees)	Sine	Cosine	Tangent
45°	.7071	.7071	1.0000
46°	.7193	.6947	1.0355
47°	.7314	.6820	1.0724
48°	.7431	.6691	1.1106
49°	.7547	.6561	1.1504
50°	.7660	.6428	1.1918
51°	.7771	.6293	1.2349
52°	.7880	.6157	1.2799
53°	.7986	.6018	1.3270
54°	.8090	.5878	1.3764
55°	.8192	.5736	1.4281
56°	.8290	.5592	1.4826
57°	.8387	.5446	1.5399
58°	.8480	.5299	1.6003
59°	.8572	.5150	1.6643
60°	.8660	.5000	1.7321
61°	.8746	.4848	1.8040
62°	.8829	.4695	1.8807
63°	.8910	.4540	1.9626
64°	.8988	.4384	2.0503
65°	.9063	.4226	2.1445
66°	.9135	.4067	2.2460
67°	.9205	.3907	2.3559
68°	.9272	.3746	2.4751
69°	.9336	.3584	2.6051
70°	.9397	.3420	2.7475
71°	.9455	.3256	2.9042
72°	.9511	.3090	3.0777
73°	.9563	.2924	3.2709
74°	.9613	.2756	3.4874
75°	.9659	.2588	3.7321
76°	.9703	.2419	4.0108
77°	.9744	.2250	4.3315
78°	.9781	.2079	4.7046
79°	.9816	.1908	5.1446
80°	.9848	.1736	5.6713
81°	.9877	.1564	6.3138
82°	.9903	.1392	7.1154
83°	.9925	.1219	8.1443
84°	.9945	.1045	9.5144
85°	.9962	.0872	11.4301
86°	.9976	.0698	14.3007
87°	.9986	.0523	19.0811
88°	.9994	.0349	28.6363
89°	.9998	.0175	57.2900
90°	1.0000	.0000	undefined

TABLE OF COTANGENTS, SECANTS, COSECANTS

Angle Degrees	Cotangent	Secant	Cosecant
0°	undefined	1.000	undefined
1°	57.2900	1.000	57.30
2°	28.6363	1.001	28.65
3°	19.0811	1.001	19.11
4°	14.3007	1.002	14.34
5°	11.4301	1.004	11.47
6°	9.5144	1.006	9.567
7°	8.1443	1.008	8.206
8°	7.1154	1.010	7.185
9°	6.3138	1.012	6.392
10°	5.6713	1.015	5.759
11°	5.1446	1.019	5.241
12°	4.7046	1.022	4.810
13°	4.3315	1.026	4.445
14°	4.0108	1.031	4.134
15°	3.7321	1.035	3.864
16°	3.4874	1.040	3.628
17°	3.2709	1.046	3.420
18°	3.0777	1.051	3.236
19°	2.9042	1.058	3.072
20°	2.7475	1.064	2.924
21°	2.6051	1.071	2.790
22°	2.4751	1.079	2.669
23°	2.3559	1.086	2.559
24°	2.2460	1.095	2.459
25°	2.1445	1.103	2.366
26°	2.0503	1.113	2.281
27°	1.9626	1.122	2.203
28°	1.8807	1.133	2.130
29°	1.8040	1.143	2.063
30°	1.7321	1.155	2.000
31°	1.6643	1.167	1.942
32°	1.6003	1.179	1.887
33°	1.5399	1.192	1.836
34°	1.4826	1.206	1.788
35°	1.4281	1.221	1.743
36°	1.3764	1.236	1.701
37°	1.3270	1.252	1.662
38°	1.2799	1.269	1.624
39°	1.2349	1.287	1.589
40°	1.1918	1.305	1.556
41°	1.1504	1.325	1.524
42°	1.1106	1.346	1.494
43°	1.0724	1.367	1.466
44°	1.0355	1.390	1.440
45°	1.0000	1.414	1.414

Angle Degrees	Cotangent	Secant	Cosecant
45°	1.0000	1.414	1.414
46°	.9657	1.440	1.390
47°	.9325	1.466	1.367
48°	.9004	1.494	1.346
49°	.8693	1.524	1.325
50°	.8391	1.556	1.305
51°	.8098	1.589	1.287
52°	.7813	1.624	1.269
53°	.7536	1.662	1.252
54°	.7265	1.701	1.236
55°	.7002	1.743	1.221
56°	.6745	1.788	1.206
57°	.6494	1.836	1.192
58°	.6249	1.887	1.179
59°	.6009	1.942	1.167
60°	.5774	2.000	1.155
61°	.5543	2.063	1.143
62°	.5317	2.130	1.133
63°	.5095	2.203	1.122
64°	.4877	2.281	1.113
65°	.4663	2.366	1.103
66°	.4452	2.459	1.095
67°	.4245	2.559	1.086
68°	.4040	2.669	1.079
69°	.3839	2.790	1.071
70°	.3640	2.924	1.064
71°	.3443	3.072	1.058
72°	.3249	3.236	1.051
73°	.3057	3.420	1.046
74°	.2867	3.628	1.040
75°	.2679	3.864	1.035
76°	.2493	4.134	1.031
77°	.2309	4.445	1.026
78°	.2126	4.810	1.022
79°	.1944	5.241	1.019
80°	.1763	5.759	1.015
81°	.1584	6.392	1.012
82°	.1405	7.185	1.010
83°	.1228	8.206	1.008
84°	.1051	9.567	1.006
85°	.0875	11.47	1.004
86°	.0699	14.34	1.002
87°	.0524	19.11	1.001
88°	.0349	28.65	1.001
89°	.0175	57.30	1.000
90°	.0000	undefined	1.000

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1 2 3	4 5 6	7 8 9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42 85 127 40 81 121	170 212 254 162 202 242	297 339 381 283 323 364
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	39 77 116 37 74 111	154 193 232 148 185 222	270 309 348 259 296 333
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	36 71 106 34 68 102	142 177 213 136 170 204	248 284 319 238 272 307
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301	33 66 98 32 63 95	131 164 197 126 158 190	229 262 295 221 253 284
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30 61 91 29 59 88	122 152 183 118 147 177	213 244 274 206 236 265
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28 57 85 28 55 83	114 142 171 110 138 165	199 228 256 193 221 248
16	20412	20683	20951	21219	21484	21748	22011	22272	22531	22789	27 53 80 26 52 78	107 134 160 104 130 156	187 214 240 182 208 233
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	26 50 76 25 49 73	101 126 151 98 122 147	176 201 227 171 196 220
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	24 48 71 23 46 69	95 119 143 93 116 139	167 190 214 162 185 208
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	23 45 68 22 44 66	90 113 135 88 110 132	158 180 203 154 176 198
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21 43 64	85 106 127	148 170 190
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20 40 60	80 100 120	140 160 180
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	20 39 58	77 97 116	135 154 174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19 37 56	74 93 111	130 148 167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18 35 53	71 89 106	124 142 159
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17 34 51	68 85 102	119 136 153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16 33 49	66 82 98	115 131 148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16 32 47	63 79 95	111 126 142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15 30 46	61 76 91	107 122 137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15 29 44	59 74 88	103 118 132
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14 29 43	57 72 86	100 114 129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14 28 41	55 69 83	97 110 124
32	50515	50650	50786	50920	51054	51188	51322	51455	51587	51720	13 27 40	54 67 80	94 107 121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13 26 39	52 65 78	91 104 117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13 25 38	50 63 76	88 101 113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12 24 37	49 61 73	85 98 110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12 24 36	48 60 71	83 95 107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12 23 35	46 58 70	81 93 104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11 23 34	45 57 68	79 90 102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11 22 33	44 55 66	77 88 99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11 22 33	44 55 66	77 88 99
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10 21 31	42 53 63	74 84 95
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10 20 31	41 51 61	71 82 92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10 20 30	40 50 60	70 80 90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10 20 29	39 49 59	68 78 88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10 19 29	38 48 57	67 76 86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9 19 28	37 47 56	65 74 84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9 18 27	36 46 55	64 73 82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9 18 27	36 45 53	63 72 81
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9 18 26	35 44 53	62 70 79

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
											1 2 3	4 5 6	7 8 9
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9 17 26	34 43 52	60 69 77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8 17 25	34 42 50	59 67 76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8 17 25	33 42 50	58 66 75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8 16 24	32 41 49	57 65 73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8 16 24	32 40 48	56 64 72
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8 16 23	31 39 47	55 63 70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8 15 23	31 39 46	54 62 69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8 15 23	30 38 45	53 60 68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7 15 22	30 37 44	52 59 67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7 15 22	29 37 44	51 58 66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7 14 22	29 36 43	50 58 65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7 14 21	28 36 43	50 57 64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7 14 21	28 35 41	48 55 62
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7 14 20	27 34 41	48 54 61
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7 13 20	27 34 40	47 54 60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7 13 20	26 33 40	46 53 59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7 13 20	26 33 39	46 52 59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6 13 19	26 32 38	45 51 58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6 13 19	25 32 38	44 50 57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6 12 19	25 31 37	43 50 56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6 12 19	25 31 37	43 50 56
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6 12 18	24 31 37	43 49 55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6 12 18	24 30 36	42 48 54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6 12 18	24 30 35	41 47 53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6 12 17	23 29 35	41 46 52
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6 12 17	23 29 35	41 46 52
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6 11 17	23 29 34	40 46 51
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6 11 17	22 28 34	39 45 50
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6 11 17	22 28 33	39 44 50
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	6 11 17	22 28 33	39 44 50
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5 11 16	22 27 32	38 43 49
81	90848	90902	90956	91009	91062	91116	91169	91222	91275	91328	5 11 16	21 27 32	37 42 48
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5 11 16	21 27 32	37 42 48
83	91908	91960	92012	92064	92117	92169	92221	92273	92324	92376	5 10 16	21 26 31	36 42 47
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	5 10 15	20 26 31	36 41 46
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5 10 15	20 26 31	36 41 46
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5 10 15	20 25 30	35 40 45
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5 10 15	20 25 30	35 40 45
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5 10 15	20 25 29	34 39 44
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5 10 15	19 24 29	34 39 44
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5 10 14	19 24 29	34 38 43
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5 9 14	19 24 28	33 38 42
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5 9 14	19 24 28	33 38 42
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	5 9 14	18 23 28	32 38 42
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5 9 14	18 23 28	32 37 42
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5 9 14	18 23 27	32 36 41
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	5 9 14	18 23 27	32 36 41
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4 9 13	18 22 27	31 36 40
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4 9 13	18 22 26	31 35 40
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4 9 13	17 22 26	31 35 39

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
											1 2 3	4 5 6	7 8 9
.00	10000	10023	10046	10069	10093	10116	10139	10162	10186	10209	2 5 7	9 12 14	16 19 21
.01	10233	10257	10280	10304	10328	10351	10375	10399	10423	10447	2 5 7	10 12 14	17 19 21
.02	10471	10495	10520	10544	10568	10593	10617	10641	10666	10691	2 5 7	10 12 15	17 20 22
.03	10715	10740	10765	10789	10814	10839	10864	10889	10914	10940	3 5 8	10 13 15	18 20 23
.04	10965	10990	11015	11041	11066	11092	11117	11143	11169	11194	3 5 8	10 13 15	18 20 23
.05	11220	11246	11272	11298	11324	11350	11376	11402	11429	11455	3 5 8	11 13 16	18 21 24
.06	11482	11508	11535	11561	11588	11614	11641	11668	11695	11722	3 5 8	11 13 16	19 21 24
.07	11749	11776	11803	11830	11858	11885	11912	11940	11967	11995	3 5 8	11 14 16	19 22 25
.08	12023	12050	12078	12106	12134	12162	12190	12218	12246	12274	3 6 8	11 14 17	20 22 25
.09	12303	12331	12359	12388	12417	12445	12474	12503	12531	12560	3 6 9	11 14 17	20 23 26
.10	12589	12618	12647	12677	12706	12735	12764	12794	12823	12853	3 6 9	12 15 18	21 24 26
.11	12882	12912	12942	12972	13002	13032	13062	13092	13122	13152	3 6 9	12 15 18	21 24 27
.12	13183	13213	13243	13274	13305	13335	13366	13397	13428	13459	3 6 9	12 15 18	21 25 28
.13	13490	13521	13552	13583	13614	13646	13677	13709	13740	13772	3 6 9	13 16 19	22 25 28
.14	13804	13836	13868	13900	13932	13964	13996	14028	14060	14093	3 6 10	13 16 19	22 26 29
.15	14125	14158	14191	14223	14256	14289	14322	14355	14388	14421	3 7 10	13 16 20	23 26 30
.16	14454	14488	14521	14555	14588	14622	14655	14689	14723	14757	3 7 10	13 17 20	24 27 30
.17	14791	14825	14859	14894	14928	14962	14997	15031	15066	15101	3 7 10	14 17 21	24 28 31
.18	15136	15171	15205	15241	15276	15311	15346	15382	15417	15453	4 7 11	14 18 21	25 28 32
.19	15488	15524	15560	15596	15631	15668	15704	15740	15776	15812	4 7 11	14 18 22	25 29 32
.20	15849	15885	15922	15959	15996	16032	16069	16106	16144	16181	4 7 11	15 18 22	26 30 33
.21	16218	16255	16293	16331	16368	16406	16444	16482	16520	16558	4 8 11	15 19 23	26 30 34
.22	16596	16634	16672	16711	16749	16788	16827	16866	16904	16943	4 8 12	15 19 23	27 31 35
.23	16982	17022	17061	17100	17140	17179	17219	17258	17298	17338	4 8 12	16 20 24	28 32 36
.24	17378	17418	17458	17498	17539	17579	17620	17660	17701	17742	4 8 12	16 20 24	28 32 36
.25	17783	17824	17865	17906	17947	17989	18030	18072	18113	18155	4 8 12	17 21 25	29 33 37
.26	18197	18239	18281	18323	18365	18408	18450	18493	18535	18578	4 8 13	17 21 25	30 34 38
.27	18621	18664	18707	18750	18793	18836	18880	18923	18967	19011	4 9 13	17 22 26	30 35 39
.28	19055	19099	19143	19187	19231	19275	19320	19364	19409	19454	4 9 13	18 22 26	31 35 40
.29	19498	19543	19588	19634	19679	19724	19770	19815	19861	19907	5 9 14	18 23 27	32 36 41
.30	19953	19999	20045	20091	20137	20184	20230	20277	20324	20370	5 9 14	19 23 28	32 37 42
.31	20417	20464	20512	20559	20606	20654	20701	20749	20797	20845	5 10 14	19 24 29	33 38 43
.32	20893	20941	20989	21038	21086	21135	21184	21232	21281	21330	5 10 15	19 24 29	34 39 44
.33	21380	21429	21478	21528	21577	21627	21677	21727	21777	21827	5 10 15	20 25 30	35 40 45
.34	21878	21928	21979	22029	22080	22131	22182	22233	22284	22336	5 10 15	20 25 31	36 41 46
.35	22387	22439	22491	22542	22594	22646	22699	22751	22803	22856	5 10 16	21 26 31	37 42 47
.36	22909	22961	23014	23067	23121	23174	23227	23281	23336	23388	5 11 16	21 27 32	37 43 48
.37	23442	23496	23550	23605	23659	23714	23768	23823	23878	23933	5 11 16	22 27 33	38 44 49
.38	23988	24044	24099	24155	24210	24266	24322	24378	24434	24491	6 11 17	22 28 34	39 45 50
.39	24547	24604	24660	24717	24774	24831	24889	24946	25003	25061	6 11 17	23 29 34	40 46 51
.40	25119	25177	25236	25293	25351	25410	25468	25527	25586	25645	6 12 18	23 29 35	41 47 53
.41	25704	25763	25823	25882	25942	26002	26062	26122	26182	26242	6 12 18	24 30 36	42 48 54
.42	26303	26363	26424	26485	26546	26607	26669	26730	26792	26853	6 12 18	24 31 37	43 49 55
.43	26915	26977	27040	27102	27164	27227	27290	27353	27416	27479	6 13 19	25 31 38	44 50 56
.44	27542	27606	27669	27733	27797	27861	27925	27990	28054	28119	6 13 19	26 32 39	45 51 58
.45	28184	28249	28314	28379	28445	28510	28576	28642	28708	28774	7 13 20	26 33 39	46 52 59
.46	28840	28907	28973	29040	29107	29174	29242	29309	29376	29444	7 13 20	27 34 40	47 54 60
.47	29512	29580	29648	29717	29785	29854	29923	29992	30061	30130	7 14 21	28 34 41	48 55 62
.48	30200	30269	30339	30409	30479	30549	30620	30690	30761	30832	7 14 21	28 35 42	49 56 63
.49	30903	30974	31046	31117	31189	31261	31333	31405	31477	31550	7 14 22	29 36 43	50 58 65

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
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.50	31623	31696	31769	31842	31916	31989	32063	32137	32211	32285	7 15 22	29 37 44	52 59 66
.51	32359	32434	32509	32584	32659	32735	32809	32885	32961	33037	8 15 23	30 38 45	53 60 68
.52	33113	33189	33266	33343	33420	33497	33574	33651	33729	33806	8 15 23	31 39 46	54 62 69
.53	33884	33963	34041	34119	34198	34277	34356	34435	34514	34594	8 16 24	32 40 47	55 63 71
.54	34674	34754	34834	34914	34995	35075	35156	35237	35318	35400	8 16 24	32 40 48	56 65 73
.55	35481	35563	35645	35727	35810	35892	35975	36058	36141	36224	8 16 25	33 41 50	58 66 74
.56	36308	36392	36475	36559	36644	36728	36813	36898	36983	37068	8 17 25	34 42 51	59 68 76
.57	37154	37239	37325	37411	37497	37584	37670	37757	37844	37931	9 17 26	35 43 52	61 69 78
.58	38019	38107	38194	38282	38371	38459	38548	38637	38726	38815	9 18 27	35 44 53	62 71 80
.59	38905	38994	39084	39174	39264	39355	39446	39537	39628	39719	9 18 27	36 45 54	63 72 82
.60	39811	39902	39994	40087	40179	40272	40365	40458	40551	40644	9 19 28	37 46 56	65 74 83
.61	40738	40832	40926	41020	41115	41210	41305	41400	41495	41591	9 19 28	38 47 57	66 76 85
.62	41687	41783	41879	41976	42073	42170	42267	42364	42462	42560	10 19 29	39 49 58	68 78 87
.63	42658	42756	42855	42954	43053	43152	43251	43351	43451	43551	10 20 30	40 50 60	70 80 89
.64	43652	43752	43853	43954	44055	44157	44259	44361	44463	44566	10 20 30	41 51 61	71 81 91
.65	44668	44771	44875	44978	45082	45186	45290	45394	45499	45604	10 21 31	42 52 62	73 83 94
.66	45709	45814	45920	46026	46132	46238	46345	46452	46559	46666	11 21 32	43 53 64	75 85 96
.67	46774	46881	46989	47098	47206	47315	47424	47534	47643	47753	11 22 33	44 54 65	76 87 98
.68	47863	47973	48084	48195	48306	48417	48529	48641	48753	48865	11 22 33	45 56 67	78 89 100
.69	48978	49091	49204	49317	49431	49545	49659	49774	49888	50003	11 23 34	46 57 68	80 91 103
.70	50119	50234	50350	50466	50582	50699	50816	50933	51050	51168	12 23 35	47 58 70	82 93 105
.71	51286	51404	51523	51642	51761	51880	52000	52119	52240	52360	12 24 36	48 60 72	84 96 108
.72	52481	52602	52723	52845	52966	53088	53211	53333	53456	53580	12 24 37	49 61 73	85 98 110
.73	53703	53827	53951	54075	54200	54325	54450	54576	54702	54828	13 25 38	50 63 75	88 100 113
.74	54954	55081	55208	55336	55463	55590	55719	55847	55976	56105	13 26 38	51 64 77	90 102 115
.75	56234	56364	56494	56624	56754	56885	57016	57148	57280	57412	13 26 39	52 66 79	92 105 118
.76	57544	57677	57810	57943	58076	58210	58345	58479	58614	58749	13 27 40	54 67 80	94 107 121
.77	58884	59020	59156	59293	59429	59566	59704	59841	59979	60117	14 27 41	55 69 82	96 110 123
.78	60256	60395	60534	60674	60814	60954	61094	61235	61376	61518	14 28 42	56 70 84	98 112 126
.79	61659	61802	61944	62087	62230	62373	62517	62661	62806	62951	14 29 43	58 72 86	101 115 130
.80	63096	63241	63387	63533	63680	63826	63973	64121	64269	64417	15 29 44	59 74 88	103 118 132
.81	64565	64714	64863	65013	65163	65313	65464	65615	65766	65917	15 30 45	60 75 90	105 120 135
.82	66069	66222	66374	66527	66681	66834	66988	67143	67298	67453	15 31 46	62 77 92	108 123 139
.83	67608	67764	67920	68077	68234	68391	68549	68707	68865	69024	16 32 47	63 79 95	110 126 142
.84	69183	69343	69503	69663	69823	69984	70146	70307	70469	70632	16 32 48	64 81 97	113 129 145
.85	70795	70958	71121	71285	71450	71614	71779	71945	72111	72277	17 33 50	66 83 99	116 132 149
.86	72444	72611	72778	72946	73114	73282	73451	73621	73790	73961	17 34 51	68 85 101	118 135 152
.87	74131	74302	74473	74645	74817	74989	75162	75336	75509	75683	17 35 52	69 87 104	121 138 156
.88	75858	76033	76208	76384	76560	76736	76913	77090	77268	77446	18 35 53	71 89 107	125 142 159
.89	77625	77804	77983	78163	78343	78524	78705	78886	79068	79250	18 36 54	72 91 109	127 145 163
.90	79433	79616	79799	79983	80168	80353	80538	80724	80910	81096	19 37 56	74 93 111	130 148 167
.91	81283	81470	81658	81846	82035	82224	82414	82604	82794	82985	19 38 57	76 95 113	132 151 170
.92	83176	83368	83560	83753	83946	84140	84333	84528	84723	84918	19 39 58	78 97 116	136 155 175
.93	85114	85310	85507	85704	85901	86099	86298	86497	86696	86896	20 40 60	79 99 119	139 158 178
.94	87096	87297	87498	87700	87902	88105	88308	88512	88716	88920	20 41 61	81 102 122	142 162 183
.95	89125	89331	89536	89743	89950	90157	90365	90573	90782	90991	21 42 62	83 104 125	146 166 187
.96	91201	91411	91622	91833	92045	92257	92470	92683	92897	93111	21 42 64	85 106 127	149 170 191
.97	93325	93541	93756	93972	94189	94406	94624	94842	95060	95280	22 43 65	87 109 130	152 174 195
.98	95499	95719	95940	96161	96383	96605	96828	97051	97275	97499	22 44 67	89 111 133	155 178 200
.99	97724	97949	98175	98401	98628	98855	99083	99312	99541	99770	23 46 68	91 114 137	160 182 205

PRESENT VALUE OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	1.01000	1.01250	1.01500	1.01750	1.02000	1.02500	1.03000	1.03500	1.04000	1.04500	1.05000	1.06000	1.07000	1.08000	1.09000	1.10000	1
2	1.02010	1.02516	1.03023	1.03531	1.04040	1.05063	1.06090	1.07123	1.08160	1.09203	1.10250	1.12360	1.14490	1.16640	1.18810	1.21000	2
3	1.03030	1.03797	1.04568	1.05342	1.06121	1.07689	1.09273	1.10872	1.12486	1.14117	1.15763	1.19102	1.22504	1.25971	1.29503	1.33100	3
4	1.04060	1.05094	1.06136	1.07185	1.08243	1.10381	1.12551	1.14752	1.16986	1.19252	1.21551	1.26248	1.31080	1.36049	1.41158	1.46410	4
5	1.05101	1.06408	1.07728	1.09062	1.10408	1.13141	1.15927	1.18769	1.21665	1.24618	1.27628	1.33823	1.40255	1.46933	1.53862	1.61051	5
6	1.06152	1.07738	1.09344	1.10970	1.12616	1.15969	1.19405	1.22926	1.26532	1.30226	1.34010	1.41852	1.50073	1.58687	1.67710	1.77156	6
7	1.07214	1.09085	1.10985	1.12912	1.14869	1.18869	1.22987	1.27228	1.31593	1.36086	1.40710	1.50363	1.60578	1.71382	1.82804	1.94872	7
8	1.08286	1.10449	1.12649	1.14888	1.17166	1.21840	1.26677	1.31681	1.36857	1.42210	1.47746	1.59385	1.71819	1.85093	1.99256	2.14359	8
9	1.09369	1.11829	1.14339	1.16899	1.19509	1.24886	1.30477	1.36290	1.42331	1.48610	1.55133	1.68948	1.83846	1.99900	2.17189	2.35795	9
10	1.10462	1.13227	1.16054	1.18944	1.21899	1.28008	1.34392	1.41060	1.48024	1.55297	1.62889	1.79085	1.96715	2.15893	2.36736	2.59374	10
11	1.11567	1.14642	1.17795	1.21026	1.24337	1.31209	1.38423	1.45997	1.53945	1.62285	1.71034	1.89830	2.10485	2.33164	2.58043	2.85312	11
12	1.12683	1.16075	1.19562	1.23144	1.26824	1.34489	1.42576	1.51107	1.60103	1.69588	1.79586	2.01220	2.25219	2.51817	2.81266	3.13843	12
13	1.13809	1.17526	1.21355	1.25299	1.29361	1.37851	1.46853	1.56396	1.66507	1.77220	1.88565	2.13293	2.40985	2.71962	3.06580	3.45227	13
14	1.14947	1.18995	1.23176	1.27492	1.31948	1.41297	1.51259	1.61869	1.73168	1.85194	1.97993	2.26090	2.57853	2.93719	3.34173	3.79750	14
15	1.16097	1.20483	1.25023	1.29723	1.34587	1.44830	1.55797	1.67535	1.80094	1.93528	2.07893	2.39656	2.75903	3.17217	3.64248	4.17725	15
16	1.17258	1.21989	1.26899	1.31993	1.37279	1.48451	1.60471	1.73399	1.87298	2.02237	2.18287	2.54035	2.95126	3.42594	3.97031	4.59497	16
17	1.18430	1.23514	1.28802	1.34303	1.40024	1.52162	1.65285	1.79468	1.94790	2.11338	2.29202	2.69277	3.15882	3.70002	4.32763	5.05447	17
18	1.19615	1.25058	1.30734	1.36653	1.42825	1.55966	1.70243	1.85749	2.02582	2.20848	2.40662	2.85434	3.37993	3.99602	4.71712	5.55992	18
19	1.20811	1.26621	1.32695	1.39044	1.45681	1.59865	1.75351	1.92250	2.10685	2.30786	2.52695	3.02560	3.61653	4.31570	5.14166	6.11591	19
20	1.22019	1.28204	1.34686	1.41478	1.48595	1.63862	1.80611	1.98979	2.19112	2.41171	2.65330	3.20714	3.86968	4.66096	5.60441	6.72750	20
21	1.23239	1.29806	1.36706	1.43954	1.51567	1.67958	1.86029	2.05943	2.27877	2.52024	2.78596	3.39956	4.14056	5.03383	6.10881	7.40025	21
22	1.24472	1.31429	1.38756	1.46473	1.54598	1.72157	1.91610	2.13151	2.36992	2.63365	2.92526	3.60354	4.43040	5.43654	6.65860	8.14027	22
23	1.25716	1.33072	1.40838	1.49036	1.57690	1.76461	1.97359	2.20611	2.46472	2.75217	3.07152	3.81975	4.74053	5.87146	7.25787	8.95430	23
24	1.26973	1.34735	1.42950	1.51644	1.60844	1.80873	2.03279	2.28333	2.56330	2.87601	3.22510	4.04893	5.07237	6.34118	7.91108	9.84973	24
25	1.28243	1.36419	1.45095	1.54298	1.64060	1.85394	2.09378	2.36324	2.66584	3.00543	3.38635	4.29187	5.42743	6.84848	8.62308	10.83471	25
26	1.29526	1.38124	1.47271	1.56998	1.67342	1.90029	2.15659	2.44596	2.77247	3.14068	3.55567	4.54938	5.80735	7.39635	9.39916	11.91818	26
27	1.30821	1.39851	1.49480	1.59746	1.70689	1.94780	2.22129	2.53157	2.88337	3.28201	3.73346	4.82235	6.21387	7.98806	10.24508	13.10999	27
28	1.32129	1.41599	1.51722	1.62541	1.74102	1.99650	2.28793	2.62017	2.99870	3.42970	3.92013	5.11169	6.64884	8.62711	11.16714	14.42099	28
29	1.33450	1.43369	1.53998	1.65386	1.77584	2.04641	2.35657	2.71188	3.11865	3.58404	4.11614	5.41839	7.11426	9.31727	12.17218	15.86309	29
30	1.34785	1.45161	1.56308	1.68280	1.81136	2.09757	2.42726	2.80679	3.24340	3.74532	4.32194	5.74349	7.61226	10.06266	13.26768	17.44940	30
31	1.36133	1.46976	1.58653	1.71225	1.84759	2.15001	2.50008	2.90503	3.37313	3.91386	4.53804	6.08810	8.14511	10.86767	14.46177	19.19434	31
32	1.37494	1.48813	1.61032	1.74221	1.88454	2.20376	2.57508	3.00671	3.50806	4.08998	4.76494	6.45339	8.71527	11.73708	15.76333	21.11378	32
33	1.38869	1.50673	1.63448	1.77270	1.92223	2.25885	2.65234	3.11194	3.64838	4.27403	5.00319	6.84059	9.32534	12.67605	17.18203	23.22515	33
34	1.40258	1.52557	1.65900	1.80372	1.96068	2.31532	2.73191	3.22086	3.79432	4.46636	5.25335	7.25103	9.97811	13.69013	18.72841	25.54767	34
35	1.41660	1.54464	1.68388	1.83529	1.99988	2.37320	2.81386	3.33359	3.94608	4.66734	5.51601	7.68609	10.67658	14.78534	20.41397	28.10244	35
36	1.43077	1.56394	1.70914	1.86741	2.03988	2.43253	2.89827	3.45026	4.10393	4.87737	5.79181	8.14725	11.42394	15.96817	22.25122	30.91268	36
37	1.44508	1.58349	1.73478	1.90009	2.08068	2.49334	2.98522	3.57102	4.26808	5.09686	6.08140	8.63609	12.22362	17.24562	24.25383	34.00395	37
38	1.45953	1.60329	1.76080	1.93334	2.12229	2.55568	3.07478	3.69601	4.43881	5.32621	6.38547	9.15425	13.07927	18.62527	26.43668	37.40434	38
39	1.47412	1.62333	1.78721	1.96717	2.16474	2.61957	3.16702	3.82537	4.61636	5.56589	6.70475	9.70351	13.99482	20.11529	28.81598	41.14478	39
40	1.48886	1.64362	1.81402	2.00160	2.20803	2.68506	3.26203	3.95925	4.80102	5.81636	7.03998	10.28572	14.97446	21.72452	31.40942	45.25926	40

COMPOUND AMOUNT OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	0.99010	0.98765	0.98522	0.98280	0.98039	0.97561	0.97087	0.96618	0.96154	0.95694	0.95238	0.94340	0.93458	0.92593	0.91743	0.90909	1
2	0.98030	0.97546	0.97066	0.96590	0.96117	0.95181	0.94260	0.93351	0.92456	0.91573	0.90703	0.89000	0.87344	0.85734	0.84168	0.82645	2
3	0.97059	0.96342	0.95632	0.94928	0.94232	0.92860	0.91514	0.90194	0.88900	0.87630	0.86384	0.83962	0.81630	0.79383	0.77218	0.75131	3
4	0.96098	0.95152	0.94218	0.93296	0.92385	0.90595	0.88849	0.87144	0.85480	0.83856	0.82270	0.79209	0.76290	0.73503	0.70842	0.68301	4
5	0.95147	0.93978	0.92826	0.91691	0.90573	0.88385	0.86261	0.84197	0.82193	0.80245	0.78353	0.74726	0.71299	0.68058	0.64993	0.62092	5
6	0.94204	0.92817	0.91454	0.90114	0.88797	0.86230	0.83748	0.81350	0.79031	0.76790	0.74622	0.70496	0.66634	0.63017	0.59627	0.56447	6
7	0.93272	0.91672	0.90103	0.88565	0.87056	0.84127	0.81309	0.78599	0.75992	0.73483	0.71068	0.66506	0.62275	0.58349	0.54703	0.51316	7
8	0.92348	0.90540	0.88771	0.87041	0.85349	0.82075	0.78941	0.75941	0.73069	0.70319	0.67684	0.62741	0.58201	0.54027	0.50187	0.46651	8
9	0.91434	0.89422	0.87459	0.85544	0.83675	0.80073	0.76642	0.73373	0.70259	0.67290	0.64461	0.59190	0.54393	0.50025	0.46043	0.42410	9
10	0.90529	0.88318	0.86167	0.84073	0.82035	0.78120	0.74409	0.70892	0.67556	0.64393	0.61391	0.55840	0.50835	0.46319	0.42241	0.38554	10
11	0.89632	0.87228	0.84893	0.82627	0.80426	0.76214	0.72242	0.68495	0.64958	0.61620	0.58468	0.52679	0.47509	0.42888	0.38753	0.35049	11
12	0.88745	0.86151	0.83639	0.81206	0.78849	0.74356	0.70138	0.66178	0.62460	0.58966	0.55684	0.49697	0.44401	0.39711	0.35553	0.31863	12
13	0.87866	0.85087	0.82403	0.79809	0.77303	0.72542	0.68095	0.63940	0.60057	0.56427	0.53032	0.46884	0.41496	0.36770	0.32618	0.28966	13
14	0.86996	0.84037	0.81185	0.78436	0.75788	0.70773	0.66112	0.61778	0.57748	0.53997	0.50507	0.44230	0.38782	0.34046	0.29925	0.26333	14
15	0.86135	0.82999	0.79985	0.77087	0.74302	0.69047	0.64186	0.59869	0.55526	0.51672	0.48102	0.41727	0.36245	0.31524	0.27454	0.23939	15
16	0.85282	0.81975	0.78803	0.75762	0.72845	0.67363	0.62317	0.57671	0.53391	0.49447	0.45811	0.39365	0.33874	0.29189	0.25187	0.21763	16
17	0.84438	0.80963	0.77639	0.74459	0.71416	0.65720	0.60502	0.55720	0.51337	0.47318	0.43630	0.37136	0.31657	0.27027	0.23107	0.19784	17
18	0.83602	0.79963	0.76491	0.73178	0.70016	0.64117	0.58739	0.53836	0.49363	0.45280	0.41552	0.35034	0.29586	0.25025	0.21199	0.17986	18
19	0.82774	0.78976	0.75361	0.71919	0.68643	0.62553	0.57029	0.52016	0.47464	0.43330	0.39573	0.33051	0.27651	0.23171	0.19449	0.16351	19
20	0.81954	0.78001	0.74247	0.70682	0.67297	0.61027	0.55368	0.50257	0.45639	0.41464	0.37689	0.31181	0.25842	0.21455	0.17843	0.14864	20
21	0.81143	0.77038	0.73150	0.69467	0.65978	0.59539	0.53755	0.48557	0.43883	0.39679	0.35894	0.29416	0.24151	0.19866	0.16370	0.13513	21
22	0.80340	0.76087	0.72069	0.68272	0.64684	0.58087	0.52189	0.46915	0.42196	0.37970	0.34185	0.27751	0.22571	0.18394	0.15018	0.12285	22
23	0.79544	0.75147	0.71004	0.67098	0.63416	0.56670	0.50669	0.45329	0.40573	0.36335	0.32557	0.26180	0.21095	0.17032	0.13778	0.11168	23
24	0.78757	0.74220	0.69954	0.65944	0.62172	0.55288	0.49193	0.43796	0.39012	0.34770	0.31007	0.24698	0.19715	0.15770	0.12640	0.10153	24
25	0.77977	0.73303	0.68921	0.64810	0.60953	0.53939	0.47761	0.42315	0.37512	0.33273	0.29530	0.23300	0.18425	0.14602	0.11597	0.09230	25
26	0.77205	0.72398	0.67902	0.63695	0.59758	0.52624	0.46369	0.40884	0.36069	0.31840	0.28124	0.21981	0.17220	0.13520	0.10639	0.08390	26
27	0.76440	0.71505	0.66899	0.62599	0.58586	0.51340	0.45019	0.39501	0.34682	0.30469	0.26785	0.20737	0.16093	0.12519	0.09761	0.07628	27
28	0.75684	0.70622	0.65910	0.61523	0.57438	0.50088	0.43708	0.38165	0.33348	0.29157	0.25509	0.19563	0.15040	0.11591	0.08955	0.06934	28
29	0.74934	0.69750	0.64936	0.60465	0.56311	0.48866	0.42435	0.36875	0.32065	0.27902	0.24295	0.18456	0.14056	0.10733	0.08215	0.06304	29
30	0.74192	0.68889	0.63976	0.59425	0.55207	0.47674	0.41199	0.35628	0.30832	0.26700	0.23138	0.17411	0.13137	0.09938	0.07537	0.05731	30
31	0.73458	0.68004	0.63031	0.58403	0.54125	0.46512	0.39999	0.34423	0.29646	0.25550	0.22036	0.16426	0.12277	0.09202	0.06915	0.05210	31
32	0.72730	0.67198	0.62099	0.57398	0.53063	0.45377	0.38834	0.33259	0.28506	0.24450	0.20987	0.15496	0.11474	0.08520	0.06344	0.04736	32
33	0.72010	0.66369	0.61182	0.56411	0.52023	0.44270	0.37703	0.32134	0.27409	0.23397	0.19987	0.14619	0.10724	0.07889	0.05820	0.04306	33
34	0.71297	0.65549	0.60277	0.55441	0.51003	0.43191	0.36605	0.31048	0.26355	0.22390	0.19036	0.13791	0.10022	0.07305	0.05339	0.03914	34
35	0.70591	0.64740	0.59387	0.54487	0.50003	0.42137	0.35538	0.29998	0.25341	0.21425	0.18129	0.13011	0.09366	0.06763	0.04899	0.03558	35
36	0.69893	0.63941	0.58509	0.53550	0.49022	0.41109	0.34503	0.28983	0.24367	0.20503	0.17266	0.12274	0.08754	0.06263	0.04494	0.03235	36
37	0.69201	0.63151	0.57644	0.52629	0.48061	0.40107	0.33498	0.28003	0.23430	0.19620	0.16444	0.11579	0.08181	0.05799	0.04123	0.02941	37
38	0.68515	0.62372	0.56792	0.51724	0.47119	0.39129	0.32523	0.27057	0.22529	0.18775	0.15661	0.10924	0.07646	0.05369	0.03783	0.02673	38
39	0.67837	0.61602	0.55953	0.50834	0.46195	0.38174	0.31575	0.26141	0.21662	0.17967	0.14915	0.10306	0.07146	0.04971	0.03470	0.02430	39
40	0.67165	0.60841	0.55126	0.49960	0.45289	0.37243	0.30656	0.25257	0.20829	0.17193	0.14205	0.09722	0.06678	0.04603	0.03184	0.02209	40

POWERS, ROOTS, AND RECIPROCALS 1–100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3,420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

Answers

CHAPTER 1

EXERCISE 1·2, Page 4 **1.** $105.21 **2.** $2,415,615 **3.** $433,073 **4.** $342,329

EXERCISE 1·3, Pages 5-6 **1.** 485 **5.** (a) 60% (b) 53% (c) 48% **6.** (a) 40% (b) 47% (c) 52%
8. (a) No profit (b) 14% (c) 33%

EXERCISE 1·4, Page 8 **1.** 50; 50; 30 **2.** 0; 440; 550; 680; 870 **3.** 40 **4.** week ending November 5
5. 83; 24; 0 **6.** November 20-25

REVIEW EXERCISE 1, Page 14 **1.** $191,972 **2.** (a) 285

CHAPTER 2

EXERCISE 2·3, Pages 20-21 **1.** (a) Toronto Stock Exchange (b) Dec. 18 to 24 (c) 2615 (d) $11\frac{5}{8}$ (e) 11 (f) $7\frac{3}{4}$ (g) $+\frac{3}{8}$ (h) $10\frac{1}{2}$ (i) $7\frac{1}{4}$ (j) 30 shares traded (n) Bartaco (o) Allarco; Alliance
2. (a) Calgary Stock Exchange; Vancouver Stock Exchange (c) Petromine (d) 5; $5\frac{1}{2}$; 9

EXERCISE 2·4, Page 22 **1.** 21.5, 29.6, 4.71%, 3.38%; 25.7, 29.1, 3.89%, 3.44%; 5, 7.76, 20%, 12.9%; 11.4, 4.62, 8.7%, 21.6%; 48, 83, 2.1%, 1.2%; 65, 46.7, 1.51%, 2.14%; .23, .16, 437.5%, 608.7%; 13.4, 16.1, 7.47%, 6.21%; 5, 5.39, 20%, 18.55% **2.** (a) (i) 15.3 (ii) 9.2 (iii) not given (b) (i) 2.4% (ii) 4.0% (iii) 4.5% (d) (i) 15.3 (ii) not given **3.** (a) 9.84 (b) 30 (c) 62.5 **4.** Penny Stocks
5. (a) 7.89% (b) 9.3% (c) 8.7% (d) 4.76% (e) 5.5% (f) 5.2% **6.** (a) 141.6 (b) 32.5 (c) 25 (d) 20 (e) 17.78 (f) 11.1

EXERCISE 2·7, Page 33 **1.** $909.20 **2.** (a) $49 (b) $121.38 (c) $10.76 **3.** (a) $.64 (b) $.50 (c) $10
5. $123,346.83 **6.** $8628.82 **7.** Net gain = $14,989.99

EXERCISE 2·9, Page 40 **3.** $2116.41 **4.** $3012.49, $2212.93, $4103.82 **5.** (a) 60 (b) 125 (c) 99 (d) 71

EXERCISE 2·10, Page 41 **1.** 6.5%, 6.5% **2.** 4.1%, 4.7%, 5.7%, 7.3%, 7.9%, 7.7%, 8%

REVIEW EXERCISE 2, Page 42 **1.** 4.1% **3.** 5.4% **5.** Loblaws: A, 1.7%, 60.5;
B, 1.7%, 60.5, Indusmin: 2.6%, 38.5

CHAPTER 3

EXERCISE 3·1, Page 44 **1.** (b) −1, +1, −1 (c) 17, 20, 23 (d) ●●●●■, ●●●●●■, ●●●●●●■ (e) 162, 486, 1458 (f) 4, 2, 1 (g) −324, −972, −2916 (h) 1, −3, −7 (i) 2^5, 2^6, 2^7 (j) $\frac{3}{16}$, 0, $-\frac{3}{16}$ (k) $\sqrt{36}$, $\sqrt{42}$, $\sqrt{48}$ (l) .0001, .00001, .000001 (m) $1+\frac{1}{5}$, $1+\frac{1}{6}$, $1+\frac{1}{7}$ (n) T, T, T

EXERCISE 3·2, Page 45 **1.** (c), (d), (h), (j), (k) **2.** (c) +3 (d) ● (h) −4 (j) $-\frac{3}{16}$ (k) $\sqrt{6}$

EXERCISE 3·3, Pages 45-46 **1.** (a) $d = 3$ (b) $t_{12} = 35$, $t_{17} = 50$ **2.** $710, 11 terms **3.** (a) 5, 7, 9, 11, 13, 15, 17; $t_{52} = 105$ (b) 14, 11, 8, 5, 2, −1, −4; $t_{52} = -136$
(c) 14, 13, 12, 11, 10, 9, 8; $t_{52} = -33$ (d) 8, 4, 0, −4, −8, −12, −16; $t_{52} = -36$
(e) −62, −60, −58, −56, −54, −52, −50; $t_{52} = 34$
(f) −95, −90, −85, −80, −75, −70, −65; $t_{52} = 155$ (g) −16, −8, 0, 8, 16, 24, 32; $t_{52} = 40$
(h) 95, 90, 85, 80, 75, 70, 65; $t_{52} = -110$ (i) 392, 384, 376, 368, 360, 352, 344; $t_{52} = 0$
(j) 70, 80, 90, 100, 110, 120, 130; $t_{52} = 40$ **4.** (a) $t_{11} = 125$, $t_{25} = 237$ (b) $t_{11} = 14$, $t_{25} = 31\frac{1}{2}$
(c) $t_{11} = 4.1$, $t_{25} = 6.9$ (d) $t_{11} = 30$, $t_{25} = -40$ (e) $t_{11} = -14.7$, $t_{25} = -51.1$ **5.** $51, $66

6. (a) $d = -6$, $t_2 = 77$ (b) $t_1 = 111$, $t_{12} = -21$ (c) $t_{10} = 25$ (d) $t_{17} = -26.33$ (e) $d = 3.5$, $t_2 = 68.5$ (f) $d = 3.5$, $t_4 = 11.5$ (g) $d = 2$, $t_1 = 52$ (h) $d = 4$, $t_{41} = 516$, $t_{42} = 520$

EXERCISE 3·4, Pages 47-48 **1.** (a) $d = .11$ (d) $d = -1\frac{1}{4}$ (e) $d = 3x + 1$ **2.** $t_{41} = 163$ **3.** $t_1 = 11$, $t_{15} = -45$ **4.** $t_3 = 46$ **5.** $t_1 = 32$, $t_{20} = 89$ **6.** $d = 3$, $t_1 = 1$ **7.** (a) $-5, 0, 5$ (b) 40, 38, 36 (c) $-21\sqrt{2}$, $-16\sqrt{2}$, $-11\sqrt{2}$ (d) 5.08, 5.3, 5.52 **8.** (a) $t_{41} = 5\frac{1}{3}$ (b) $t_{41} = 20\frac{1}{5}$ (c) $t_{41} = 10\frac{1}{3}$ (d) $t_{41} = 6\frac{11}{12}$ **9.** $n = 42$ **10.** $n = 31$ **11.** (a) $n = 20$ (b) t_{30} (c) t_{18} **12.** (a) 11th second (b) 944 feet

EXERCISE 3·5, Pages 51-52 **1.** (a) 153 (b) 88 (c) 44 **2.** (a) 150 (b) 108 (c) 119 **3.** (a) 100 (b) 18 (c) 21 **4.** 68 inches **5.** 1152 **6.** 641,728 **7.** (a) 714 (b) 416 (c) −3888 **8.** (a) 10 (b) 12 **9.** 7500 **10.** 37,100 **11.** 21 **12.** −149, −103, −60 **13.** 1024 feet

EXERCISE 3·6, Page 54 **1.** (a) 2 (b) 3 (c) $\frac{1}{2}$ (d) −2 (e) −1 **2.** (b) −1 (e) 3 (f) $\frac{1}{2}$ (g) 3 (l) .1

EXERCISE 3·7, Page 55 **1.** (a) 64, 256 (b) 135, 405 (c) −24, 48 (d) 20, 10 (e) 1, $\frac{1}{3}$ (f) .128 (g) 9, 81 **2.** (a) −3 (b) $\frac{2}{3}$ (c) 5 (d) 20 (e) $-\frac{5}{2}$ **3.** 64 **4.** (a) 320 (b) −768 (c) $\frac{32}{81}$ (d) 16 **5.** (a) t_6 (b) t_{11} (c) t_8

EXERCISE 3·8, Pages 56-57 **1.** (a) −1 (b) −1016 (c) 381 **2.** (a) 511, 255 (b) 20,470 **3.** (a) 217 (b) $\frac{2186}{54}$ (c) 234 (d) $\frac{1023}{4}$ (e) $\frac{781}{81}$ **4.** 150 **5.** (a) 31 (b) 128 **6.** 127 **7.** $10,485.75 **8.** 9331 **9.** $\frac{1023}{4}$ **10.** 484

REVIEW EXERCISE 3, Page 58 **1.** (a) −3680 (b) 13 **2.** 123, 120, 117, 114, 111 **3.** $\frac{127}{18}$

CHAPTER 4

EXERCISE 4·1, Page 60 **1.** (a) $70 (b) $166.50 (c) $35.00 **2.** (a) $160 (b) $2160 (c) $2160 (d) $172.80 (e) $2332.80 (f) $186.62 (g) $2519.42 **3.** (a) $480 (b) $2480 (c) $39.42 less **4.** (a) 2 (b) $24 (c) $24 (e) $624 (f) $648.96 after 1 year, $674.92 after $1\frac{1}{2}$ years **5.** (a) .05 (b) .015 (c) .025

EXERCISE 4·2, Page 62 **1.** (a) $i = .06$; $n = 5$ (b) $i = .02$; $n = 24$ (c) $i = .02$; $n = 20$ **2.** (a) $133.82 (b) $1280.82 **3.** $1921.24 **4.** $1425.76 **5.** $788.89 **6.** $2971.90

EXERCISE 4·3, Page 63 **2.** (i) .66506 (ii) .62460 (iii) .68058 (iv) .49447 (v) .37689 (vi) .76491 **3.** $2665 **4.** (a) $2811 (b) $801.00 (c) $298 **5.** $3151 **6.** $11,168 **7.** $3545 **8.** (a) $18,622 (b) $9267

EXERCISE 4·4, Page 65 **2.** (a) (i) 8% (ii) 7% (b) (i) $500 $(1.04)^{19}$; $500 (ii) $100 $(1.035)^{15}$; $100 (c) (i) $r = 1.04$; 20 (ii) $r = 1.035$; 16

EXERCISE 4·5, Page 66 **1.** (a) (i) 1.34010 (ii) 1.31080 (iii) 1.80611 (iv) 2.15893 (v) 1.34489 (vi) 1.67535 **2.** (a) (i) $A = \$500 + \$500(1.04) + \$500(1.04)^2 + \$500(1.04)^3 + \$500(1.04)^4$ (ii) $t_1 = \$500$, $r = 1.04$, $n = 5$ (iii) $2708.13 (b) (i) $A = \$300 + \$300(1.03) + \$300(1.03)^2 + \$300(1.03)^3$ (ii) $t_1 = \$300$, $r = 1.03$, $n = 4$ (iii) $1255.10 **3.** $6349.65 **4.** $49,194.43 **5.** $3685.70 **6.** $96,435.47

EXERCISE 4·6, Page 68 **1.** (a) (i) $P = \$100(1.06)^{-3} + \$100(1.06)^{-2} + \$100(1.06)^{-1}$ (ii) $t_1 = \$100(1.06)^{-3}$, $r = 1.06$, $n = 3$ (iii) $2673 (b) (i) $P = \$400(1.05)^{-4} + \$400(1.05)^{-3} + \$400(1.05)^{-2} + \$400(1.05)^{-1}$ (ii) $t_1 = \$400(1.05)^{-4}$, $r = 1.05$, $n = 4$ (iii) $1418.40 (c) (i) $P = \$1000(1.06)^{-3} + \$2000(1.06)^{-2} + \$2000(1.06)^{-1} + \$1000$ (iii) $5506.42 (d) (i) $P = \$1000(1.03)^{-6} + \$1000(1.03)^{-5} + \$1000(1.03)^{-4} + \$1000(1.03)^{-3} + \$1000(1.03)^{-2} + \$1000(1.03)^{-1}$ (ii) $t_1 = \$1000(1.03)^{-6}$, $r = 1.03$, $n = 6$ (iii) $5417.33 **2.** $6177.44 **3.** $1877 **4.** $12,977.50 **5.** $59,509.33

EXERCISE 4·7, Page 70 **1.** (ii) $\$16{,}000 = t_1 + t_1(1.08) + t_1(1.08)^2$ (iv) $4158.41 (iii) $A = \$16{,}000$, $t_1 = x$, $r = 1.08$, $n = 3$ **2.** (ii) $\$4000 = t_1 + t_1(1.06) + t_1(1.06)^2 + t_1(1.06)^3 + t_1(1.06)^4$ (iv) $t_1 = \$768.74$ (iii) $A = \$4000$, $t_1 = x$, $r = 1.06$, $n = 5$ **3.** (ii) $\$8000 = x(1.07)^{-4} + x(1.07)^{-3} + x(1.07)^{-2} + x(1.07)^{-1}$ (iii) $A = \$8000$, $t_1 = x(1.07)^{-4}$, $r = 1.07$, $n = 4$ (iv) $2361.87 **4.** (ii) $\$15{,}000 = x(1.06)^{-5} + x(1.06)^{-4} + x(1.06)^{-3} + x(1.06)^{-2} + x(1.06)^{-1}$ (iii) $A = \$15{,}000$, $t_1 = x(1.06)^{-5}$, $r = 1.06$, $n = 5$ (iv) $3560.97 **5.** (a) (ii) $\$4000 = x(1.04)^{-8} + x(1.04)^{-7} + \ldots + x(1.04)^{-1}$ (iii) $A = \$4000$,

$t_1 = x(1.04)$, $r = 1.04$, $n = 8$ (iv) \$594.11 (b) (ii) $\$12{,}000 = x(1.0125)^{-15} + x(1.0125)^{-14} + \ldots + x(1.0125)^{-1}$ (iii) $A = \$12{,}000$, $t_1 = x(1.0125)^{-15}$, $r = 1.0125$, $n = 15$ (iv) $x = \$882.31$
6. (a) (ii) $\$3000 = x(1.035)^{-6} + x(1.035)^{-5} + \ldots + x(1.035)^{-1}$ (iii) $A = \$3000$, $t_1 = x(1.035)^{-6}$, $r = 1.035$, $n = 6$ (iv) \$563.00 (b) (ii) $\$7000 = x(1.02)^{-16} + x(1.02)^{-15} + \ldots + x(1.02)^{-1}$ (iii) $A = \$7000$, $t_1 = x(1.02)^{-16}$, $r = 1.02$, $n = 16$ (iv) \$515.55 **7.** (ii) $\$25{,}000 = x + x(1.025) + \quad . + x(1.025)^{17}$ (iii) $A = \$25{,}000$, $t_1 = x$, $r = 1.025$, $n = 18$ (iv) \$1116.75

EXERCISE 4·9, Page 72 **1.** (a) \$25,090.56 (b) \$8794.12 **2.** (a) \$9726.50 (b) \$29,441.87 **3.** (a) \$12,385.46 (b) \$23,197.50

EXERCISE 4·10, Page 72 **1.** \$16,987 **2.** \$10,790.81

REVIEW EXERCISE 4, Page 74 **1.** (a) \$1120 (b) \$3467.98 **2.** \$1603.80 **3.** \$22,584.45 **4.** \$6883.37 **5.** \$2022.72 **6.** \$1547.22

CHAPTER 5

EXERCISE 5·1, Page 77 **1.** Rationals: $\sqrt{25}, \sqrt[3]{8}, \sqrt[4]{81}, \sqrt[3]{-27}, \sqrt[4]{16}, \sqrt[3]{-125}, \sqrt{144}, \sqrt[5]{243}, \sqrt[4]{625}$. Irrationals: $\sqrt{80}, \sqrt{125}, \sqrt[3]{12}, \sqrt[5]{125}, \sqrt[3]{36}, \sqrt{50}$

EXERCISE 5·2, Page 78 **1.** (a) 0 (b) $5\sqrt{5}$ (c) $3\sqrt{2}$ (d) $19\sqrt{7}$ (e) $11\sqrt{6} - 6\sqrt{5}$ (f) $6\sqrt{2} - \sqrt{3}$ (g) $4\sqrt{7} - 16\sqrt{5}$ **2.** (a) 18 (b) 42 (c) 30 (d) 70 (e) 84 (f) 195 (g) 1232 (h) 3 (i) 4 (j) 4
3. (a) $6\sqrt{21}$ (b) $240\sqrt{15}$ (c) $80\sqrt{6}$ (d) $324\sqrt{5}$ (e) $\sqrt{\frac{13}{2}}$ (f) $4\sqrt{5}$ (g) $\frac{15}{\sqrt{3}}$ (h) $4\sqrt{5}$

EXERCISE 5·3, Page 79 **1.** (a) $\frac{2\sqrt{5}}{5}$ (b) $\sqrt{6}$ (c) $2\sqrt{7}$ (d) $\frac{4\sqrt{3}}{3}$ (e) $\frac{\sqrt{6}}{6}$ (f) $\frac{\sqrt{15}}{6}$ (g) $\frac{\sqrt{6}}{5}$ (h) 5
2. (a) $5 - 2\sqrt{6}$ (b) 1 (c) $3 + \sqrt{6} - \sqrt{15} - \sqrt{10}$ (d) -2 (e) $12 + 2\sqrt{35}$ (f) -2 (g) $a + 2\sqrt{ab} + b$ (h) $a - b$ (I) $\sqrt{7} - \sqrt{8}, 2\sqrt{5} + \sqrt{8}, \sqrt{3} + 2, 4 - \sqrt{5}, a + \sqrt{b}, \sqrt{a} - 5, \quad 3\sqrt{x} + 2\sqrt{y}, a\sqrt{w} - \sqrt{m}$
3. (a) $5(2 + \sqrt{3})$ (b) $\frac{8}{3}(\sqrt{5} - \sqrt{2})$ (c) $-2(\sqrt{6} + 3)$ (d) $\frac{21 + 5\sqrt{15}}{2}$ (e) $\frac{42 - 16\sqrt{6}}{3}$ (f) $\frac{76 + 18\sqrt{35}}{39}$

EXERCISE 5·4, Page 81 **1.** (a) $n = 49$ (b) $x = 9$ (c) $x = 10$ (d) $z = 6$ (e) $n = 3$ (f) $z = 3$ (g) $x = 8$ (h) $x = 10$ **2.** (a) 2, 4 (b) 3, −1 (c) 7, −2 **3.** (a) {7} (b) {0,4} (c) {9} (d) {1,9} (e) {36} (f) {8,2} (g) {2,5} (h) {11,71} (i) {3,19} (j) {1,4}

EXERCISE 5·5, Page 82 **1.** a^{12} **2.** 2^7 **3.** 2^{27} **4.** 3^{20} **5.** $21b^{10}$ **6.** $24z^9$ **7.** $4x$ **8.** $3a$ **9.** $4b^4$ **10.** $125y^4$ **11.** $27k^7$ **12.** $16x^{15}$ **13.** 3 **14.** a^3b **15.** 81 **16.** 1 **17.** −1 **18.** $\frac{-4b^3}{a^2}$ **19.** b^{x^2} **20.** −1 **21.** $\frac{81k^4}{2m^3}$ **22.** $\frac{1}{2^x}$ **23.** $\frac{1}{5^{2a}}$ **24.** $\frac{m^3n^3 - 1}{n^3}$

EXERCISE 5·6, Page 83 **1.** (a) $\sqrt{x}$ (b) $\sqrt[3]{64}$ (c) $\frac{1}{\sqrt[3]{9}}$ (d) $\sqrt[4]{7}$ (e) $\sqrt{m}$ (f) $\sqrt[9]{a^{15}}$ (g) $\frac{1}{\sqrt{a}}$ (h) $\sqrt{x^3}$ **2.** (a) 16 (b) 25 (c) 9 (d) 27 (e) $\frac{2}{5}$ (f) −8 (g) 32 (h) $\frac{1}{3\sqrt[3]{9}}$ (i) $\frac{1}{4}$ (j) $\frac{1}{9}$ (k) $\frac{1}{4}$ (l) $20x^3$ **3.** (a) 3 (b) $4\sqrt[12]{4}$ (c) $\sqrt[30]{a}$ (d) 20 (e) 1 (f) $\sqrt{x}$

EXERCISE 5·7, Page 84 **1.** (a) $\log_2 8 = 3$ (b) $\log_5 25 = 2$ (c) $\log_9 81 = 2$ (d) $\log_4 16 = 2$ (e) $\log_{10} 10{,}000 = 4$ (f) $\log_2 128 = 7$ (g) $\log_5 125 = a$ (h) $\log_8 64 = n$ (i) $\log_3 p = m$ (j) $\log_x y = a$
2. (a) $100{,}000 = 10^5$ (b) $243 = 3^5$ (c) $343 = 3^7$ (d) $625 = 5^4$ (e) $6561 = 3^8$ (f) $1024 = 4^5$ (g) $32 = 2^x$ (h) $64 = 8^n$ (i) $k = m^h$ (j) $m = b^a$

EXERCISE 5·8, Page 86 **1.** (d) 10^0, 2×10^0, 4×10^0, 5×10^0, 7×10^0, 8×10^0, 10^1, 2×10^1, 4×10^1, 5×10^1, 7×10^1, 8×10^1 **2.** (b) $3.5 = 10^{.54}$ (c) $6.5 = 10^{.81}$ (d) $8.5 = 10^{.93}$ (e) $5.5 = 10^{.74}$

(f) $8.8 = 10^{.94}$ (g) $1.5 = 10^{.17}$ (h) $43 = 10^{1.65}$ (i) $56 = 10^{1.75}$ (j) $38 = 10^{1.6}$ (k) $63 = 10^{1.8}$ (l) $78 = 10^{1.9}$
3. (b) $\log_{10}3.5 = .54$ (c) $\log_{10}6.5 = .81$ (d) $\log_{10}8.5 = .93$
(e) $\log_{10}5.5 = .74$ (f) $\log_{10}8.8 = .94$ (g) $\log_{10}1.5 = .17$ (h) $\log_{10}43 = 1.65$ (i) $\log_{10}56 = 1.75$
(j) $\log_{10}38 = 1.6$ (k) $\log_{10}63 = 1.8$ (l) $\log_{10}78 = 1.9$ **4.** (a) 1.25 (b) 1.45 (c) 1.55 (d) .64
(e) .59 (f) .75 (g) 1.85 (h) 1.82 **5.** (a) .2 (b) .2 (c) .62 (d) .62 (e) .88 (f) .88 **6.** (a) 0 (b) 1
(c) 0 (d) 1 (e) 0 (f) 1

EXERCISE 5·9, Page 87 **1.** (a) 8700 (b) 2250 (c) 2600 (d) 2600 (e) 46,500 (f) 5550 **2.** (a) 21.5
(b) 5 (c) 1.9 (d) 7.9 (e) 160 (f) 5.35

EXERCISE 5·10, Page 88 **2.** (a) .38 (b) .86 (c) .77 (d) .97 (e) .83 (f) .6 **3.** (a) 1 (b) 2 (c) 5 (d) −3
(e) −6 (f) −1 **4.** (a) .5 + 2 (b) .88 + 1 (c) .64 + 3 (d) .79 + 4 (e) .93 + 5 (f) .76 + 5 (g) .57 − 1
(h) .77 − 4 (i) .7 − 2 (j) .43 − 6 **5.** (a) 20 (b) 2.45 (c) 3500 (d) 690 (e) .55 (f) .000485 (g) 25.5
(h) 1500 (i) 6,600,000 (j) .0000415

EXERCISE 5·11, Page 90 **1.** (a) $x = \log_4 y$ (b) $x = \log_9 y$ (c) $x = \log y$ **2.** (a) $y = 3^x$ (b) $y = 8^x$ (c) $y = m^x$
3. (a) 2 (b) 5 (c) 20 (d) 445

EXERCISE 5·12, Page 92 **1.** (a) .5185 + 1 (b) .5378 + 2 (c) .9325 + 2 (d) .8859 + 2 (e) .9209 + 1
(f) .8826 + 1 (g) .8414 + 1 (h) .8833 + 1 (i) .4989 − 2 (j) .8816 − 1 (k) .6201 − 2 (l) .7782 − 3
2. (a) 17.26 (b) 518.2 (c) 2195 (d) .02096 (e) 2763 (f) .03769 (g) .3273 (h) .3084 (i) 369.7
(j) .0005812 (k) .8798 (l) 30,020 **3.** (a) 25,000 (b) 2600 (c) 39,500 (d) 300 (e) 9,777,000
(f) 6,724,000

EXERCISE 5·13, Page 94 **2.** (a) 2500 (b) 3,940,000 (c) 2.0 (d) .13 (e) 540 (f) 7.1 (g) 5800
(h) 85,000 (i) 11 (j) 11 (k) 13 (l) 130,000 (m) 1400 (n) .0005 **3.** (a) 460 (b) 370 (c) 52 (d) 35.0
(e) 110 (f) 700 **5.** (a) 7200 (b) .0605 (c) 57.08 (d) 15 (e) 26.2 (f) 17.8 (g) 9.06 (h) 28 (i) 25
6. 140,800 **7.** .71 **8.** .9262 **9.** 39 **10.** 782 **11.** .6126 **12.** 6.1291 **13.** 3.095 **14.** .4007 **15.** 31
16. 88 **17.** 15.23 **18.** 651 **19.** 1.95

REVIEW EXERCISE 5, Page 95 **1.** (a) $\frac{\sqrt{6}-\sqrt{2}}{2}$ (b) $-\frac{7}{5}$ **2.** (a) 27, 3 (b) 0, 15 **3.** (a) $\frac{1}{5}$ (b) 3^{7n}
(c) 1 **4.** (a) 3,580,000 (b) 251 (c) 2.01 (d) 5300

CHAPTER 6

EXERCISE 6·1, Page 100 **1.** (i) .866 (ii) 1.732 (iii) .5 (iv) .667 (v) .767 (vi) 1.21
(vii) $\frac{t}{r}$ (viii) $\frac{t}{s}$ (ix) $\frac{s}{r}$ (x) .577 (xi) .767 (xii) $\frac{s}{t}$ **2.** (a) $\frac{171}{200}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{4}{5}$ (e) $\frac{99}{10\sqrt{2}}$

EXERCISE 6·2, Pages 101-102 **1.** 6.4 **2.** 38 **3.** $m\angle B = 14°$, $b = 13$, $c = 56$ **4.** $n\angle A = 33°$, $b = 33$,
$c = 39$ **5.** 112 yards **6.** 53.6 feet **7.** 376 feet **8.** 297 feet **9.** $\sin 20° = .342$, $\cos 50° = .642$,
$\tan 25° = .465$, $\sin 35° = .574$, $\cos 25° = .91$, $\tan 37° = .756$, $\sin 65° = .882$, $\cos 58° = .53$,
$\tan 28° = .555$

EXERCISE 6·3, Page 105 **1.** (a) *I* (b) *III* (c) *IV* (d) *IV* (e) *II* (f) *IV* (g) *IV* **2.** (a) − (b) − (c) + (d) −
(e) − (f) − (g) − (h) − (i) + **3.** (a) 1. (b) −.7071 (c) −.5 (d) −.866 (e) −.5774 (f) −1. (g) −.866
(h) −.5 (i) .6428 (j) 1. (k) .9659 (l) −.2126 (m) −.866 (n) .766 (o) −.5

EXERCISE 6·4, Page 107 **1.** (a) II (b) I, III (c) $\sin\theta$ (d) III (e) III, IV (f) 3 (g) $\sin\theta$, $\tan\theta$
4. $\sin\theta = \frac{4}{5}$, $\tan\theta = -\frac{4}{3}$ **5.** $\csc\theta = \frac{13}{12}$, $\sec\theta = \frac{13}{5}$ **6.** $\cos\theta = -\frac{1}{2}$, $\csc\theta = -\frac{2}{\sqrt{3}}$
7. $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = -1$ **8.** $-\frac{24}{7}$ **9.** 1

EXERCISE 6·5, Pages 109-110 **1.** (a) $m\angle A = 35°$, $m\angle c = 85°$, $c = 115$ (b) $m\angle P = 54°$, $p = 332$,
$q = 310$ (c) $m\angle Z = 65°$, $x = 327$, $y = 180$ **2.** 55°, 390, 35° **3.** 215 **4.** 1785×10^6 miles **5.** 206, 267

6. 3×10^6 miles **7.** 2565×10^6 miles **8.** 2663×10^6 miles

REVIEW EXERCISE 6, Page 112 **1.** 5410 miles **2.** (a) −.5 (b) −2.1445 (c) 2.0 (d) .7002 (e) −.5 (f) −1.0642 **3.** $\sin\theta = \frac{12}{13}$, $\csc\theta = \frac{13}{12}$, $\cot\theta = \frac{5}{12}$, $\cos\theta = \frac{5}{13}$, $\sec\theta = \frac{13}{5}$ **4.** $\sin\theta = -\frac{3}{5}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$ **5.** 2694.9 **6.** 69.9

CHAPTER 7

EXERCISE 7·1, Pages 118-120 **1.** (a) [(3, 1), (7, 4)] (b) [(5, −3), (2, −1)] (c) [(−6, −2), (−3, −2)] (d) [(5, −8), (−2, −5)] (e) [(7, −6), (−9, −3)] **2.** (a) (−2, −9) (b) (−4, −16) (c) (6, −11) (d) (8, −4) (e) (−14, 20) (f) (25, −48) **3.** (a) (6, 7) (b) (6, −6) (c) (1, 4) (d) (6, 4) (e) (0, 0) (f) (0, 0) **4.** (12, 1), (7, 8), (0, 0) **5.** (a) (2, 1) (b) (13, 3) (c) (22, 15) (d) (−3, 3) (e) (−4, −5) (f) $(d - m, f - n)$ **7.** (a) (14, 12) (b) (−2, −12) (c) (27, −18) (d) (41, −16) **8.** (a) $(ha + ma, hb + mb)$ (b) $(2nc, 2nd)$ (d) $(me + ma, mf + mb)$ (e) $(sg + sb, sh + sd)$ **10.** (a) (14, −14) (b) (24, −6) (c) (−21, 30) (d) (−53, 30) **11.** (−15, 30) **12.** (a) (10, 5) (b) (7, −4) (c) (−9, −4) (d) (3, −13) (e) (−2, −9) **13.** (a) (2, 4) (b) (−3, −5) (c) (1, 5) (d) (−4, 4) (e) (2, 6) (f) (−1, −2) (g) (−3, −3) (h) (−3, 0) (i) $(\frac{13}{2}, -4)$ (j) $(-2, -\frac{2}{5})$ **14.** (a) 78 pounds, 47° to the horizontal (b) 40.9 pounds; 28.7 pounds **15.** (a) (2, 6) (b) (2, 6) **16.** (a) (8, 14) (b) (8, 14) **17.** (8, 6), 1000 **18.** 250 mph northwest

EXERCISE 7·2, Pages 122-123 **2.** (a) 492 (b) 738 (c) 532 (d) 648 **3.** 260; 212 **4.** 3000; 1732 **5.** (a) 331 (b) 361 (c) 563 (d) $x = 92$

EXERCISE 7·3, Page 124 **1.** 6.8, 9.8 **2.** 9.2, 11.2 **3.** 30.5 mph; 58.5° to the bank **4.** 54 **5.** 92, 143 **6.** 227

EXERCISE 7·4, Page 126 **1.** 440 **2.** $a = 15.2$, $b = 26.1$, $e = 35$, $f = 34.3$, $m = 20.4$, $p = 25.1$ **3.** 94°; 80°; 135° **4.** 60; 79; 84 **5.** 3000; 1732 **6.** 260; 212

EXERCISE 7·5, Pages 127-128 **1.** (a) $-\cos 80°$ (b) $-\cos 33°$ (c) $-\cos 81°$ **2.** (a) $-\cos 68° = .3746$ (b) $-\cos 43° = .7314$ (c) $-\cos 18° = .9511$ (d) $\sin 80° = .9848$ (e) $\sin 33° = .5446$ (f) $\sin 37° = .6018$ **3.** $t = 17.6$, $m\angle P = 26°$, $m\angle S = 31°$; $m = 7.2$, $m\angle P = 74°$, $m\angle N = 36°$; $r = 8.7$, $m\angle T = 62°$, $m\angle S = 78°$ **4.** 34°, 44°, 102° **5.** 49°, 59°, 72° **6.** 10 newtons **7.** 50 newtons, 36° to $\vec{v}$, 44° to $\vec{u}$

EXERCISE 7·6, Pages 130-131 **1.** (a) no; $6\frac{1}{7}$ pound force to 51 pound force (b) no; $1\frac{4}{7}$ pound force to 550 pound force (c) no; 8 pound force to 56 pound force (d) no; $\frac{2}{3}$ pound force to 26 pound force **2.** (a) 113.6 (b) 68.2 (c) 101 (d) 50.7 (e) 1.03 (f) 1129.4 **3.** 6.5 **4.** (a) 45.5 (b) 120

EXERCISE 7·7, Pages 133-134 **1.** 150, 2 **5.** (a) 4.13 (b) 2.85 **6.** 82.2% **7.** 97.7% **8.** 40% **9.** (a) 937 (b) .44 (c) .43 (d) 97%

EXERCISE 7·8, Page 138 **1.** (7, −2, 15), (11, −8, −23) **2.** yes; yes; yes **3.** Line: all; plane: all; space: all **4.** (a) [(0, 0, 0), (3, 4, 5)] (b) [(0, 0, 0), (10, 9, 13)] (c) [(3, 4, 5), (10, 9, 13)]

EXERCISE 7·9, Page 140 **1.** 187 **2.** (a) 1.7 (b) 13.6 (c) 12.7 (d) 9.7 (e) 3.9 (f) 4.2 **3.** 700 mph **4.** (a) (9, −4, 15) (b) (9, −4, 15) (c) (13, 3, −16) (d) (13, 3, −16) **5.** (a) (5, 1, 2) (b) (5, 1, 2) (c) (8, 0, −1) (d) (8, 0, −1) **6.** (−3, 7, −13) **7.** (a) (12, −4, −8) (b) (−12, −4, −8) (c) (3, −3, 21) (d) (−3, 3, −21) no

REVIEW EXERCISE 7, Page 141 **1.** (6, −4) **2.** 674.7 **3.** 737.3 **4.** 46.4 **5.** .167 **6.** 13.6 **7.** horizontal: 7.5; vertical: 13

CHAPTER 8

EXERCISE 8·1, Page 145 **1.** $ax + ay$ **2.** $6a^2 - 8ab$ **3.** $6x + 21y - 12z$ **4.** $-30m - 6m^2 + 8m^3$

5. $15x^2 + 20xy^2$ **6.** $a^2 + 3ab + b^2$ **7.** $3p^2 - rp - 10r^2$ **8.** $6x^2 - 17xy + 10y^2$
9. $5x^2 + 10x^2y - 8xy - 3xy^2 + 3y^2$ **10.** $8a^2b - 6a^2b^2 + 4ab^2 + 4a - 3ab + 2b$ **11.** $2xy^3 - 3x^3y - 5x^2y^2$
12. $a^3 - 27b^3$ **13.** $8m^3 + 125n^3$ **14.** $27x^3 - 64y^3$ **15.** $a^4 - 2a^2b^2 + b^4$ **16.** $125 - 100x + 40x^2 - 8x^3$
17. $9a^4 - 6a^3b + 13a^2b^2 - 4ab^3 + 4b^4$ **18.** $m^4 - 2m^2n^2 + n^4$

EXERCISE 8·2, Page 146 **1.** $\frac{3ab}{2}$ **2.** $\frac{3xz}{2}$ **3.** $4x - 5$ **4.** $4xy + 3x - 5y$ **5.** $7m^2n + 4mp^2 - 5n^2p$ **6.** $a - 5$
7. $3a + 4b$ **8.** $4m^2 + 2mn + n^2$ **9.** $9x^2 + 3x + 1$ **10.** $16a^2 - 20ab + 25b^2$ **11.** $8x^3 - 4x^2y + 2xy^2 + y^3$
12. $3a^2 + 2b$ **13.** $a - b$ **14.** $4x^2 - 9y^2$ **15.** $m^2 + 4m - 3$ **16.** $4a^2 - a$

EXERCISE 8·3, Page 148 **1.** $(x - 3)(x - 1)$ **2.** $(x - 3)(x + 2)$ **3.** $(x + 4)(x - 1)$ **4.** $(x - 7)(x - 2)$
5. $(x + 5)(x + 4)$ **6.** $(x + 6)(x - 5)$ **7.** $(x + 8)(x - 2)$ **8.** $(x - 12)(x + 1)$ **9.** $(x - 12)(x + 2)$
10. $(x - 12)(x - 2)$ **12.** $(x - 9)^2$ **13.** $(a + 8)^2$ **14.** $(2y + 3)(y - 4)$ **15.** $(2b - 3)(b - 3)$ **16.** $(2x + 5)(x - 2)$
17. $(2x - 3)^2$ **18.** $(3a - 2)(a + 3)$ **19.** $(3x + 4)(x + 3)$ **20.** $(2y - 5)(2y - 3)$
21. $(2x - 7)(2x + 3)$ **22.** $(5m - 1)(m + 7)$ **23.** $(8z - 1)(3z - 8)$ **24.** $(6x - 5)(7x + 6)$
25. $(4a - 9)(5a - 7)$ **26.** $(7b + 7)(7b - 4)$ **27.** $(6y + 1)(2y - 7)$ **28.** $3(3x - 2)^2$ **29.** $(8x - 5)(3x - 5)$
30. $(5x + 7)(7x + 6)$ **31.** $(8x - 9)(5x + 6)$

EXERCISE 8·4, Page 149 **1.** (a) $\frac{m+3}{m+6}$ (b) $\frac{t+4}{t+7}$ (c) $\frac{z-6}{3z+4}$ (d) $\frac{a+4}{2a-3}$ (e) $\frac{x+7}{x+3}$ (f) $\frac{x-2}{2x-5}$
2. (a) $\frac{a-3}{a+3}$ (b) $\frac{x+1}{x-1}$ (c) $\frac{2a+1}{a+2}$ (d) $\frac{x+2}{x+3}$ (e) $\frac{b-1}{b+3}$ **3.** (a) $\frac{1}{2}$ (b) $\frac{6x-7}{3x+2}$ (c) $\frac{3b+4}{2b-6}$ (d) $\frac{2}{3}$
4. (a) 1 (b) $\frac{5x-3}{5x+3}$ (c) $\frac{5b-2}{4b+3}$

EXERCISE 8·5, Page 151 **1.** $\frac{3a-3}{a^2}$ **2.** $\frac{7a-7b}{ab}$ **3.** $\frac{2x-2y}{xy}$ **4.** $\frac{2n+6m}{mn}$ **5.** $\frac{11s-12r}{3rs}$ **6.** $19\frac{19b-3a}{6ab}$
7. $-\frac{1}{a-b}$ **8.** $-\frac{12n}{m^2-n^2}$ **9.** $\frac{2a^2+5ab-3b^2}{a^2-b^2}$ **10.** $\frac{12ar-4br-10as-5bs}{6a^2+ab-b^2}$ **11.** $\frac{3x^2+3xy+6y^2}{x^2-4y^2}$
12. $\frac{8xy+12y^2-6x+9y}{4x^2-9y^2}$ **13.** $\frac{2x^2+13xy}{6x^2+xy-2y^2}$ **14.** $\frac{4ab}{a^2-b^2}$ **15.** $\frac{4a-4b}{3ab}$ **16.** $\frac{7x^2-3xy}{30xy}$ **17.** $\frac{7}{x-y}$
18. $\frac{5a+7}{a^2+5a+6}$ **19.** $\frac{2z^2-10}{z^2+4z+3}$ **20.** $\frac{4b^2-9b-14}{b^2+2b-3}$ **21.** $-\frac{12}{2x^2-x-28}$ **22.** 0
23. $-\frac{16}{a^2+a-2}$ **24.** $\frac{1}{x+4}$ **25.** $\frac{2}{b^2+3b+2}$

EXERCISE 8·6, Page 152 **2.** $\{-2, -5\}$ **3.** $\{3, 1\}$ **4.** $\{3, -2\}$ **5.** $\{-4, -3\}$ **6.** $\{-6, -1\}$ **7.** $\{-7, 2\}$
8. $\{8, -5\}$ **9.** $\{-9, 2\}$ **10.** $\{-6\}$ **11.** $\{4\}$ **12.** $\{-\frac{3}{2}, 1\}$ **13.** $\{\frac{1}{2}, 3\}$ **14.** $\{\frac{2}{3}, -\frac{3}{2}\}$
15. $\{\frac{3}{4}, -\frac{1}{2}\}$ **16.** $\{-\frac{5}{2}, -\frac{1}{2}\}$ **17.** $\{\frac{5}{3}, -1\}$ **18.** $\{\frac{3}{4}, -\frac{3}{2}\}$ **19.** $\{-\frac{4}{3}, -7\}$ **20.** $\{-\frac{2}{3}, -\frac{9}{2}\}$ **21.** $\{-\frac{7}{2}, \frac{3}{2}\}$
22. $\{\frac{3}{2}\}$ **23.** $\{\frac{5}{4}, -\frac{5}{4}\}$ **24.** $\{-\frac{12}{5}, 3\}$ **25.** $\{\frac{1}{6}, \frac{7}{2}\}$ **26.** $\{\frac{6}{5}, -\frac{10}{3}\}$

EXERCISE 8·7, Pages 154-155 **3.** $\{2 \pm \sqrt{7}\}$ **4.** $\{4, 2\}$ **5.** $\{3, -5\}$ **6.** $\{3, -7\}$ **7.** $\{6, 4\}$ **8.** $\{7, 3\}$
9. $\{\pm\sqrt{\frac{29}{2}} - \frac{3}{2}\}$ **10.** $\{1, -6\}$ **11.** $\{-3, -6\}$ **12.** $\{-3\}$ **13.** $\{2, 1\}$ **14.** $\left\{\pm\frac{\sqrt{31}}{2} + 2\right\}$
15. $\{\frac{7}{2}, 3\}$ **16.** $\{3, -\frac{4}{3}\}$ **17.** $\{\pm\sqrt{\frac{47}{8}} - \frac{17}{8}\}$ **18.** $\{\pm\sqrt{\frac{71}{12}} - \frac{1}{2}\}$

EXERCISE 8·8, Page 156 **1.** $\{-4, -1\}$ **2.** $\{3, -2\}$ **3.** $\{-5, 2\}$ **4.** $\{-5, -3\}$ **5.** $\{4, 3\}$
6. $\{-10, 4\}$ **7.** $\{\frac{3}{2}, -4\}$ **8.** $\{\frac{4}{3}, \frac{1}{2}\}$ **9.** $\left\{\frac{-3 \pm \sqrt{-11}}{2}\right\}$, nonreal **10.** $\left\{\frac{3 \pm \sqrt{15}}{2}\right\}$
11. $\{3\}$ **12.** $\{\frac{2}{3}\}$ **13.** $\left\{\frac{-3 \pm \sqrt{-37}}{24}\right\}$, nonreal **14.** $\{\frac{7}{2}\}$ **15.** $\left\{\frac{-5 \pm \sqrt{265}}{10}\right\}$
16. $\left\{\frac{3 \pm \sqrt{-131}}{14}\right\}$, nonreal **17.** $\{\frac{1}{6}, -\frac{25}{6}\}$ **18.** $\left\{\frac{7 \pm \sqrt{-143}}{16}\right\}$, nonreal

EXERCISE 8·9, Page 158 **1.** 5″, 8″, 4″ **2.** 6″, 8″, 10″ **3.** 1 foot **4.** 6, 7 **5.** 3′ × 6′ **6.** 10
7. 9, 11 **8.** \$8, \$12

REVIEW EXERCISE 8, Page 159 **1.** $6a^3 + 7a^2b - 72b^2 - 6b^3$ **2.** $4a^2 + 6ab + 9b^2$ **3.** (a) $(a - 4)(a - 2)$
(b) $(3x - 1)(2x + 1)$ (c) $(2b - 1)(b - 5)$ (d) $(3y + 7)(4y - 3)$ **4.** (a) $\frac{x-1}{3}$ (b) $y - 1$ (c) $\frac{4a^2 - 9a - 14}{a^2 + 2a - 3}$
5. (a) $\{\frac{7}{2}, -3\}$ (b) $\{-\frac{27}{16}, -\frac{29}{16}\}$ (c) $\{\frac{4}{3}, -3\}$ **6.** $6' \times 3\frac{1}{2}'$

CHAPTER 9

EXERCISE 9·2, Pages 162-163 **1.** (a) 96 (b) 4 (c) 4 **2.** (a) 900; 7.5; 7.5 (b) 1406; 9.375; 9.375
(c) 120,000; 50; 50 (d) 128; 4; 4 **3.** 75 seconds; 80,000 feet; 150 seconds

EXERCISE 9·4, Page 169 **1.** (a) 3 (b) 2 (c) 7 (d) no **2.** (a) negative (b) $x = -3$ (c) 0 (d) $(-3, 0)$
(e) 2; $(-3, 0)$, $(0, -9)$ (f) yes **3.** (i) (a) $(0, 0)$, $(5, 0)$ (b) $x = \frac{5}{2}$ (c) $\frac{-25}{4}$, min (d) $(0, 0)$
(ii) (a) $(3, 0)$, $(-2, 0)$ (b) $x = \frac{1}{2}$ (c) $\frac{25}{2}$, max (d) $(0, 6)$ (iii) (a) $(-5, 0)$ (b) $x = -5$ (c) 0, min (d) $(0, 25)$
(iv) (a) $(\frac{5}{2}, 0)$, $(-2, 0)$ (b) $x = \frac{1}{4}$ (c) $-\frac{81}{8}$, min (d) $(0, -10)$ (v) (a) $(-5, 0)$, $(3, 0)$ (b) $x = -1$
(c) −16, min (d) $(0, -15)$ (vi) (a) $(-\frac{8}{3}, 0)$, $(2, 0)$ (b) $x = -\frac{1}{3}$ (c) $-\frac{49}{3}$, min (d) $(0, -16)$
(vii) (a) imaginary (b) $x = 2$ (c) 12, min (d) $(0, 16)$ (viii) (a) $(\frac{1}{5}, 0)$, $(-5, 0)$ (b) $x = -\frac{12}{5}$
(c) $-\frac{169}{5}$, min (d) $(0, -5)$

EXERCISE 9·5, Pages 171-172 **1.** $4\frac{1}{2}' \times 4\frac{1}{2}'$; $20\frac{1}{4}$ sq ft **2.** 18 sq ft; $3' \times 6'$ **3.** 7¢ **4.** $5\frac{1}{2}$, $5\frac{1}{2}$
5. 13″ × 13″; 169 **6.** −12; 2 **7.** 1. (a) 225′ (b) 7.5 sec; 2. 3078 sq ft 3. (a) $7\frac{1}{2}$ (b) \$1343.75 (c) 250

REVIEW EXERCISE 9, Page 173 **1.** (b) no (c) $(-\frac{2}{3}, 0)$, $(-\frac{5}{2}, 0)$, $-\frac{121}{24}$ (d) $x = -\frac{19}{12}$ **2.** 294′
(b) 3.5 sec (c) 7 sec **3.** $16' \times 24'$

CHAPTER 10

EXERCISE 10·4, Pages 182-183 **1.** (i) $F(6, 0)$, $x = -6$ (ii) $F(-4, 0)$, $x = 4$ (iii) $F(9, 0)$, $x = -9$
(iv) $F(-18, 0)$, $x = 18$ **2.** $y^2 = -20x$ **3.** $y^2 = 27x$, no **4.** $x^2 = 12y$ **5.** (a) $y^2 = 16x$ (b) $y^2 = 6x$
(c) $x^2 = \frac{8}{3}y$ (d) $y^2 = \frac{125}{3}x$

EXERCISE 10·5, Page 184 **1.** (a) $(3, 9)$, $(-1, 1)$ (b) $(-9, 3)$ (c) no solution (d) $(3, 0)$, $(-2, 10)$
3. (a) $(12, 12)$, $(3, -6)$ (b) $(0, 0)$, $(3, -1)$ (c) $(4, 1)$ (d) $(-3, 0)$ (e) $(0, -12)$, $(4, 8)$
(f) $(3, 2)$, $(-1, 14)$ (g) $(3, -15)$ (h) $(\frac{8}{5}, \frac{4}{5})$, $(8, -4)$ (i) no solution (j) no solution

EXERCISE 10·6, Page 185-186 **1.** $x^2 = -9y$ **2.** $x^2 = -\frac{250}{4}y$ **3.** $x^2 = 32y$; $x^2 = -80y$ **4.** $y^2 = \frac{72}{5}x$
5. $A = 5.6$, $B = 5.0$, $C = 11.4$ **6.** yes **7.** $y^2 = \frac{1681}{5}x$

REVIEW EXERCISE 10, Page 188 **1.** (a) $F(5, 0)$, $x = -5$ (b) $F(-8, 0)$, $x = 8$ (c) $F(\frac{1}{8}, 0)$, $x = -\frac{1}{8}$
2. $F(0, 6)$, $y = 0$, $V(0, 0)$, $y = -6$ **3.** $y^2 = -4x$ **4.** $x^2 = -5y$ **5.** (a) $y^2 = -28x$ (b) $x^2 = -16x$ (c) $y^2 = 8x$
6. (a) $(4, 8)$, $(1, -4)$ (b) $(6, 5)$, $(-6, 1)$

CHAPTER 11

EXERCISE 11·4, Page 196 **2.** (a) (i) $(0, -1)$, $(0, 1)$ (ii) 4 (iii) $2\sqrt{5}$ (b) (i) $(-9, 0)$, $(9, 0)$
(ii) 24 (iii) 30 (c) (i) $(-5, 0)$, $(5, 0)$ (ii) 24 (iii) 26 (d) (i) $(-4, 0)$, $(4, 0)$
(ii) 16 (iii) 20 **3.** (a) $\frac{x^2}{81} + \frac{y^2}{225} = 1$ (b) $\frac{x^2}{41} + \frac{y^2}{16} = 1$ (c) $\frac{x^2}{25} + \frac{y^2}{169} = 1$ (d) $\frac{x^2}{400} + \frac{y^2}{144} = 1$

(e) $\frac{x^2}{100}+\frac{y^2}{64}=1$ (f) $\frac{x^2}{9}+\frac{y^2}{13}=1$ **5.** $x=y, x=-y$ (a) (0, 0), centre (b) 4, yes

6. (0, – 12), (0, 12) **7.** (– 8, 0), (8, 0) **8.** *A*, *C*, *E* **9.** $\frac{x^2}{169}+\frac{y^2}{144}=1$ (b) $\frac{x^2}{216}+\frac{y^2}{225}=1$

EXERCISE 11·6, Pages 202-203 **2.** (a) (i) – 4, 4 (ii) (– 5, 0), (5, 0) (iii) 6 (iv) 8 (v) 5
(b) (i) – 8, 8 (ii) (– 10, 0), (10, 0) (iii) 12 (iv) 16 (v) 10 (c) (i) – 12, 12 (ii) (0, – 13), (0, 13)
(iii) 10 (iv) 24 (c) 13 (d) (i) – 2, 2 (ii) ($\sqrt{13}$, 0), ($-\sqrt{13}$, 0) (iii) 6 (iv)4 (v) $\sqrt{13}$
3. (a) $\frac{x^2}{25}-\frac{y^2}{64}=1$ (b) $\frac{x^2}{16}-\frac{y^2}{41}=-1$ (c)$\frac{x^2}{49}-\frac{y^2}{16}=-1$ (d) $\frac{x^2}{144}-\frac{y^2}{25}=1$ (e)$\frac{x^2}{9}-\frac{y^2}{64}=1$

5. (b) (3, – 6) (e) $\frac{x^2}{9}-\frac{y^2}{36}=1; \frac{x^2}{9}-\frac{y^2}{36}=-1$

EXERCISE 11·7, Page 204 **1.** 2. (a) 8 (b) 16 (c) 24 (d) 4; 3. (a) 10 (b) $2\sqrt{41}$
(c) 8 (d) 24 (e) 6

EXERCISE 11·8, Page 206 **2.** (a) (4, – 2) (b) (– 3, 0), (0, – 8) (c) (– 5, – 4), (5, 4)
(d) (9, 8), (– 12, – 6) (e) no solution (f) no solution **3.** (a) (6, 4), (– 6, – 4) (b) (0, 5), (3, 0)
(c) (3, 2), (2, 0) (d) (3, 5) (e) (– 6, 0), (2, 4) (f) (2, 4) (g) no solution (h) no solution
4. $xy=256; xy=20$ **5.** (a) 2 (b) 2 **6.** no

EXERCISE 11·9, Pages 206-207 **1.** (a) $x^2y=k$ (b) 1024; $x^2y=1024$ (c) 24.1 **3.** $xy \approx 320$

REVIEW EXERCISE 11, Page 208 **1.** (a) (i) (–12, 0), (12, 0) (ii) (0, 2), (0, –2) (iii) (–11.8, 0), (11.8, 0)
(iv) 24 (v) 4 (vi) 24 (b) (i) (–9, 0), (9, 0) (ii) (0, 10), (0, –10) (iii) (0, 4.4), (0, –4.4) (iv) 20
(v) 18 (vi) 20 **2.** (i) $\frac{4x^2}{169}+\frac{4y^2}{69}=1$ (ii) $\frac{x^2}{81}+\frac{y^2}{144}=1$ (iii) $\frac{x^2}{49}+\frac{y^2}{121}=1$ **3.** no **4.** (a) (i) –12, 12
(ii) (–20, 0), (20, 0) (iii) 24 (iv) 32 (v) 20 (vi) 24 (b) (i) 10, –10 (ii) (0, 12.6), (0, –12.6) (iii) 20
(iv) 48 (v) 12.6 (vi) 20 **5.** (a) no solution (b) (–6, 0), (10, 4)

CHAPTER 12

EXERCISE 12·3, Pages 212-215 **1.** (a) 1 (b) –1 (c) –270°, 90° (d) –90°, 270°
(e) –360°, –180°, 0°, 180°, 360° (f) –330°, –210°, 30°, 150°; –150°, –30°, 210°, 330°
(g) every 360° (h) 360° **2.** (a) 1; –360°, 360°; maximum (b) –180°, 180° (c) 1 (d) 360°
(e) Curves identical in shape but out of phase by 90° **3.** (a) 90° (b) Move cos θ right by 90°.
(c) $\cos\theta=\sin(\theta+90°)$ (d) yes (e) no (f) no (g) no **4.** (a) no (b) no, –90° (c) 1, –135°
(d) No, because tan θ becomes infinitely large near $\pm 90°$. ∴ amplitude is undefined.
(e) 180°, half of period for sin θ and cos θ **5.** (c) no, 1 (d) 180°, half.

6. (b) same amplitude; period of cos 2θ is 180° **7.** (b) $\frac{1}{b}\times 360°$ (c) yes **8.** (b) amplitude = *a*
(c) 3, 180°; $\frac{1}{2}$, 120°; 2, 720° (d) *a*, $\frac{1}{b}\times 360°$ **9.** (a) 360°, 1 (b) 1440° (c) $\frac{5}{2}$ (d) 1440°, $\frac{5}{2}$ **10.** (a) 360°, 1
(b) 360° (c) 1 (d) 1080° (e) 2 (f) 1080°, 2 **11.** (a) 180° (b) 90° (e) 270°

REVIEW EXERCISE 12, Page 217 **1.** (d) 0.50, 0.87, 0.50, 0.50, –0.82, 0.77, –0.34, 0.17, 0.71
2. (a) increasing (b) maximum (c) 0 (d) minimum **3.** (a) increasing (b) increasing (c) 0
(d) increasing **4.** (a) out of phase by 90° (b) period of sin 2θ is half that of sin θ
(c) amplitudes: 1, 3 **5.** (a) 1, 360° (b) undefined, 180° (c) 2, 360° (d) undefined, 720° (e) 1, 120°
(f) 1, 1440° (g) 2, 1080° (h) $\frac{1}{2}$, 120°

CHAPTER 13

EXERCISE 13·4, Page 224 **2.** $157.50 **3.** (a) $190.34 (b) $158.84 (c) $118.52 (d) $229.16

4. (a) \$202.60 (b) \$67,780 (c) \$34,780 **5.** (a) \$122.34 (b) \$33,861.60 (c) \$15,111.60

EXERCISE 13·6, Page 227 **1.** \$207 **2.** \$187 **3.** \$163.26

REVIEW EXERCISE 13, Page 228 **2.** \$825 **3.** (a) \$163.05 (b) \$52,915 (c) \$28,415 **4.** \$206.46

CHAPTER 14

EXERCISE 14·1, Page 233 **1.** (a) \$48, \$72, \$17.28 (b) \$326, \$391.20 (c) \$61.25, \$9 (d) \$12, \$18, \$7.20 **5.** \$119.84 **6.** \$5.76 **7.** (a) \$38.70, \$38.70 (b) \$74.35, \$78.81 (c) \$184.26, \$199 (d) \$410.30, \$430.80 (e) \$3929.96, \$3972.01 (f) \$75, \$89.04 (g) \$79.50, \$93.45 (h) \$80.36, \$90.00 (i) \$52.05, \$58.30 **8.** \$36.57

EXERCISE 14·2, Pages 236-237 **4.** (a) \$4260.60 (b) \$4729.26 **5.** \$3562.50; \$0; \$652.50; \$637.50; \$1080; \$0; \$111.60; \$665; \$875 **6.** (b) BMW; \$808.08 **7.** (a) \$812.50, \$1072.00, \$716.00 (b) 0, \$160.90, \$166.00 (c) 0, \$1618, 0 (d) \$4062.50, \$3902.90, \$2672.00 (e) \$487.50, \$468.35, \$320.64 (f) \$4550, \$4371.25, \$2992.64 **8.** \$10,301.04 **9.** \$2196 **10.** \$118.14 **11.** 24%, 40%, 36% **13.** \$137.38 **15.** \$18.20 **17.** yes

EXERCISE 14·3, Page 239 **2.** (a) £116.96 (b) 3595.50 francs (c) 756.01 crowns (d) 61,538.46 yen (e) \$325.87 (f) 280.33 cruzeiros (g) 1100.24 pesos (h) \$84.08 **3.** (a) \$8.79 (b) \$12.57 (c) \$7.74 (d) \$3.89 (e) \$63.18 (f) \$5.06 (g) \$10.06 (h) \$31.71 **4.** (a)

EXERCISE 14·4, Page 241 **2.** \$298.51 **3.** \$904.90 **4.** 13,043.48 francs **5.** (a) 0.628 marks (b) 0.304 kroners (c) 0.240 zloty (d) 2.047 kroners

REVIEW EXERCISE 14, Page 242 **1.** \$2462.75 **2.** \$3.97 **3.** 6896.55 schillings, 1036.27 francs, 775.19 Swiss francs, 650.41 marks

CHAPTER 15

EXERCISE 15·2, Pages 246-247 **1.** (a) 120, 1961 (b) 1965 (c) 170, 240 (d) 1964 and 1970 **2.** (a) 250, 250 (b) 27, 73 (c) 50

EXERCISE 15·3, Page 247 **1.** 74.7 **2.** 28.8 **3.** 59.5

EXERCISE 15·4, Page 249 **1.** 14.2 **2.** 36 **3.** 29.5 **4.** 14 **5.** \$2 **6.** 35 **7.** 18 **8.** $5' \frac{1}{2}''$

EXERCISE 15·5, Page 250 **1.** (a) 52 (midpoint of class of greatest frequency) (b) 50 (c) 73 **2.** (a) 25 (b) 17 (c) 13 (d) \$1

EXERCISE 15·6, Page 252 **1.** $\frac{2}{5}$ **2.** $\frac{1}{3}$ **3.** 3 **4.** $\frac{2}{5}$

EXERCISE 15·7, Pages 255-256 **2.** (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{5}{6}$ **3.** $\frac{3}{7}$ **4.** (a) $\frac{1}{13}$ (b) $\frac{3}{13}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$ **5.** $\frac{3}{7}$ **6.** (a) $\frac{1}{5}$ (b) $\frac{1}{2}$ **7.** (a) $\frac{1}{11}$ (b) $\frac{14}{33}$ **8.** (b) $\frac{1}{2}$ **9.** (b) $\frac{1}{4}$ (c) $\frac{3}{8}$ **10.** (a) 36 (b) 2, $\frac{1}{18}$ (c) $\frac{1}{6}$ (d) $\frac{1}{6}$ (e) $\frac{1}{6}$ (f) $\frac{1}{6}$ **11.** (a) $\frac{25}{102}$ (b) $\frac{1}{221}$ (c) $\frac{13}{51}$ (d) $\frac{11}{221}$ **12.** (a) $\frac{33}{16,660}$ (b) $\frac{1}{54,145}$ **13.** $\frac{1}{10}$

EXERCISE 15·8, Page 257 **1.** (a) 16 (b) (i) 6 (ii) 4 (iii) 1 (iv) 11 (c) (i) $\frac{1}{4}$ (ii) $\frac{1}{16}$ (iii) $\frac{15}{16}$ **2.** (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$ **3.** (a) $\frac{1}{4}$ (b) $\frac{11}{16}$ (c) $\frac{1}{4}$ (d) $\frac{1}{4}$ (f) $\frac{3}{8}$ (g) $\frac{5}{16}$

EXERCISE 15·9, Pages 260-261 **1.** (a) 36 (b) 6 **2.** (a) $\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{6}$ (d) $\frac{1}{18}$ (e) $\frac{1}{36}$ (f) $\frac{11}{36}$ (g) $\frac{5}{12}$ (h) $\frac{1}{9}$ (i) $\frac{5}{6}$ **3.** (a) $\frac{1}{20}$ (b) $\frac{7}{100}$ (c) $\frac{3}{4}$ **4.** (i) $\frac{1}{2}$ (ii) $\frac{7}{8}$ (iii) 1 (iv) $\frac{3}{8}$ **5.** (a) $\frac{3}{13}$ (b) $\frac{1}{2}$ (iii) $\frac{8}{13}$ (iv) $\frac{3}{26}$ **6.** (i) $\frac{3}{20}$ (ii) $\frac{3}{5}$ (iii) $\frac{2}{5}$ (iv) $\frac{23}{100}$ (v) $\frac{11}{50}$ **7.** (i) $\frac{7}{30}$ (ii) $\frac{13}{30}$ (iii) $\frac{1}{3}$ **8.** (i) $\frac{1}{2}$ (ii) $\frac{43}{90}$ (iii) $\frac{4}{5}$ (iv) $\frac{1}{5}$

REVIEW EXERCISE 15, Page 263 **1.** (c) 17.7, 18, 18 **2.** 39.8, 39.5, 34.5 **3.** (a) $\frac{3}{26}$ (b) $\frac{7}{26}$ (with ace low) (c) $\frac{2}{13}$ (d) $\frac{2}{13}$ (e) $\frac{3}{13}$ **4.** $\frac{1}{4}, \frac{39}{248}$ **5.** (a) $\frac{2}{15}$ (b) $\frac{13}{15}$ (c) $\frac{2}{15}$ (d) $\frac{1}{3}$ (e) $\frac{2}{5}$

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